Top-Down Models for Credit Derivatives

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Abstract

In this thesis we derive and implement models, called generalized default counting process models, to price credit derivatives. Derived models use the default point process of the derivative underlying to determine the default probabilities. This class of models is very flexible and can be used in the pricing of single-name and multi-name credit derivatives. In case of multi-name credit derivatives, the models can be implemented as a top-down or bottom-up models. This means that we can either choose to model the complete underlying as a single entity or constituent firms separately. In this thesis, we focus on the top-down models.

In addition to model derivation, we discuss and implement efficient model calibration schemes. The calibration is formulated either as a minimization problem or a root-solving exercise, where the model implied prices match the market observed quotes of the calibration products. We propose gradient based methods to address minimization problems, and use bisection to address the root-solving problems.

At the end of this thesis we provide example pricings where we price CDS index tranches and CDS index options. In CDS index tranche pricing, we compare how the model implied prices match the market observed quotes when the model is calibrated with the whole index. In CDS index option pricing, we focus on the model implied risk numbers, i.e., how the option prices change based on the changes in credit risk (flat spread) and volatility. In the example pricings we use single-factor models that can be calibrated with a straightforward root-solving exercise.

The motivation to consider new class of models to price credit derivatives stems from the challenges of the existing models. The current market standard is to use copula-framework which has various deficiencies especially in the case of multi-name products. These deficiencies include the arbitrary modelling choices, calibration challenges and unnecessary complexity. The generalized default counting process models considered in this thesis aim to overcome these deficiencies.

Keywords Credit derivatives, generalized default counting process models, top-down models

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Tiivistelmä

Tässä työssä kehitetään luottoriskijohdannaisten hinnoitteluun tarkoitettuja malleja, jotka perustuvat yleistettyihin laskuriprosesseihin. Johdetut mallit määrittävät luottovastuutapahtumien todennäköisyyden luottovastuutapahtumien pistelaskuriprosessin avulla. Työssä kehitetyt mallit ovat hyvin joustavia ja pystyvät hinnoittelemaan luottoriskijohdannaisia, jotka seuraavat joko yhden yhtiön tai usean yhtiön luottoriskiä.

Työssä käydään läpi mallien implementointi tietokoneella mallien johtamisen lisäksi. Lisäksi työssä käsitellään mallien tehokasta kalibrointia. Kalibrointiongelma kirjoitetaan joko minimointiongelmana tai funktion nollakohdan ratkaisuongelmana siten, että mallin antamat hinnat vastaavat markkinoilla oleviin hintoihin. Työssä käytetään gradienttiin perustuvia menetelmiä minimointiongelmien ratkaisuun, ja puolitushakua funktion nollakohdan määrittämiseen.

Työ sisältää esimerkki hinnoitteluja, joissa hinnoitellaan CDS indeksituotteita ja CDS optioita. CDS indeksituotteiden hinnoittelussa mallin antamia arvoja verrataan markkinoilta saatuihin hintoihin tapauksessa, jossa malli on kalibroitu koko indeksillä. CDS optioiden hinnoittelussa keskitytään mallin antamiin riskilukuihin, eli kuinka option hintaan vaikuttavat luottoriskin ja volatiliteetin muutos. Esimerkki hinnoitteluissa käytetään yhden muuttujan mallia, jossa kalibrointi on funktion nollakohdan ratkaisuongelma.

Tämän työn tavoitteena on käsitellä uusia luottoriskijohdannaisten hinnoitteluun soveltuvia malleja, jotka pystyvät ratkaisemaan nykyisten markkinastandardin mukaisten mallien ongelmia. Tällä hetkellä markkinastandardina on käyttää copula-funktioihin perustuvia malleja, joihin liittyy useita ongelmia etenkin johdannaisissa, joiden kohde-etuuteen kuuluu useita yhtiöitä. Tässä työssä käsiteltävät laskuriprosessimallit pyrkivät ratkaisemaan nämä ongelmat.

Avainsanat Luottoriskijohdannaiset, yleistetyt pistelaskuriprosessit

Contents

1 Introduction

In the last two decades, credit derivatives have revolutionized the way market participants trade on the credit risk. From the invention of credit default swaps (CDS) contracts in 1994 [1], the size of the credit derivatives market has exploded. Before the financial crisis credit derivatives market grew exponentially, which was due to the fact that market participants, mainly banks, insurers and hedge funds, had a strong interest to trade on the credit risk. In the banks this interest mainly stemmed from the need of hedging the credit risks in the balance sheet, while in the insurance companies and hedge funds credit derivatives were mainly used as a tool to seek profits on the credit risk.

After the phase of rapid growth in the first decade of the 21st century, the credit derivatives market has consolidated. In Figure 1, the data of total global notional amount of outstanding CDS contracts is shown from the second half of 2004 to the beginning of 2018, collected by BIS Statistics. The current outstanding notional amount of outstanding CDS contracts is around 8 to 10 trillion US dollars, which is same as in 2004 and 2005.

Even though the credit derivatives market consists of various derivatives products, such as credit-linked notes (CLNs), credit spread options (CSOs) and swaptions, the most important credit derivatives products are CDS contracts. The popularity of CDS contracts comes from the fact that they are simple and effective contracts to trade on credit risk. In essence, CDS contract is a bilateral agreement in which the credit risk of reference entity is transferred from one counterparty to the other for compensation called premium payment. In case of credit event in the reference entity, the payer of premium payment, known as protection buyer, receives a payment from premium receiver, known as protection seller. Therefore, CDS contract can be seen as a credit event insurance on the reference entity.

The reference entity of CDS contract is usually a firm or a basket of firms. The CDS contract having a single firm as reference entity is known as single-name CDS, while the CDS contract having multiple firms as reference entities is knows as multi-name CDS. Currently, the most popular multi-name CDS contracts are so called CDS index contracts, in which the reference entity of the CDS contract is an index of firms picked according to the credit ratings of the firms. Examples of such CDS index contracts are CDX High Yield, CDX Investment Grade, iTraxx Crossover, and iTraxx Europe CDS indexes provided by IHS Markit. The CDX High Yield and iTraxx Crossover indexes are collections of 100 American high yield companies and 75 European high yield companies, respectively. The CDX Investment Grade and iTraxx Europe indexes are collections of 125 American investment grade companies and 125 European investment grade companies, respectively. The reference entity lists of these indexes are updated semiannually, on roll dates, and the inclusion and exclusion of firms in the index updates on roll dates is determined by the Liquidity Lists created

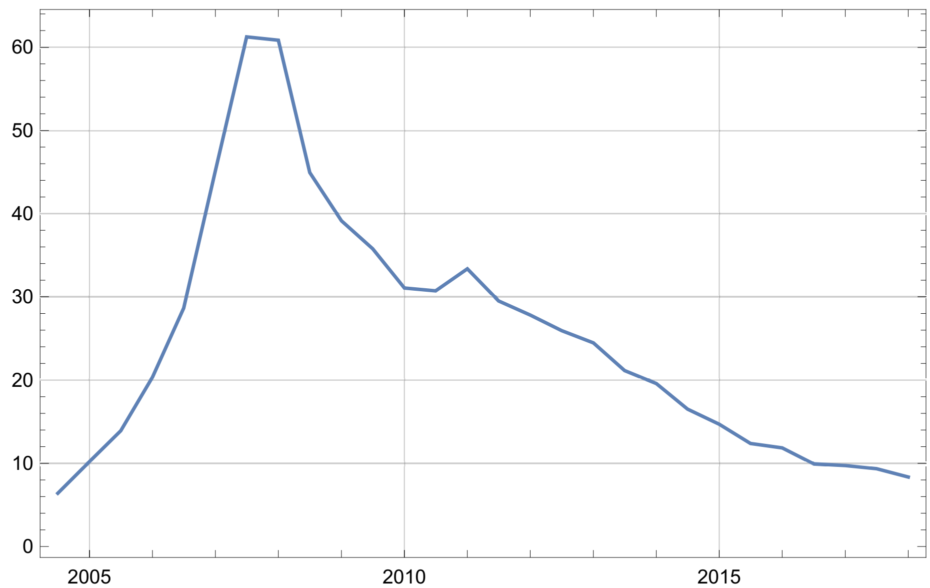

Figure 1: The total notional amount of outstanding CDS contracts in USD trillions. Source: BIS Statistics.

by Markit [2], [3]. These four indexes trade on 1 year, 3 year, 5 year, 7 year and 10 year contracts, with 5 year and 10 year contracts having the highest liquidity.

The advantage of CDS indexes is that they provide industry and country wide diversification with a single CDS trade. Also, the bid-ask spreads of the CDX and iTraxx indexes are very low, with high trading volumes, making it easy to trade on the CDS indexes. In addition to taking a position as protection buyer or protection seller in the CDS contract having exposure on the whole index, known as linear contract, investor can take position on a specific part of a index, known as tranche. In the linear contract, protection seller pays the protection buyer the protection payment in every credit event, while in tranche contract, protection seller pays protection buyer the protection payment only if the amount of credit events in the reference entity index is in the interval of tranche attachment and detachment points. The CDX and iTraxx indexes have highly liquid standard tranches, for example CDX HY having standard tranches of 0-15%, 15-25%, 25-35% and 35-100%. The first percentage in the tranche, known as attachment point, describes how many credit events need to occur before protection payment is paid on the next credit event, and the second percentage in the tranche, known as detachment point, describes after how many credit events protection seller is not entitled to pay protection payment to protection buyer on the next credit event. The tranche attachment points are usually denoted in percentages, and in case of CDX HY 15-25% tranche the number of companies in attachment and detachment points are 15 and 25, respectively.

As the CDS contracts have gained popularity, the need to model and price them accurately has become imminent. The pricing models have gained noticeable interest both in the academic literature and among trading professionals.

In multi-name CDS contracts, the current market standard modeling framework is so-called copula framework, in which the joint survival probability of reference entities is modeled with copula functions [4]. The survival probability denotes the probability that reference entity does not cause a credit event in the given time period. The popularity of copula framework is due to its flexibility of choosing various marginal probability densities, and the intuitive nature of the model.

Due to the shortcomings of the copula framework, including the arbitrary choice of copula functions, the challenges in the calibration of the copula model, and the problem of estimating some model parameters, such as correlation matrix of the reference entities, credit derivatives researches and professionals have started to look for models that describe the underlying default counting process of the reference entities better than the copula framework.

The set of multi-name credit derivative pricing models can be categorized into two categories. The first group of models, known as bottom-up models, aim to model the default counting process by modeling the default counting processes for each reference entity separately, and then aggregating them to get the multi-name default counting process (such as in [4], [5]). The other group of models, known as top-down models, aim to directly model the multi-name default counting process, and with that process, model the single reference entity default counting processes, if needed (such as in [6], [7]).

In this thesis, we focus on the top-down models of the default counting process. We derive a class of top-down models that have small computational complexity with a very limited amount of free parameters. The goal is that the calibration of the models is simple, and that the prices of the credit derivatives can be calculated quickly with the calibrated models. Also, the economic interpretation of the derived models should be intuitive, to make sure that the model implied values of the credit derivatives are economically justified. The models considered in this thesis belong to a class of so-called generalized default counting processes. Similar models to the models that we consider in this thesis have been derived in [6] and [8]. In the model derivation in subsection 3.2 we rederive the model proposed in [6].

The choice of focusing on the low-complexity top-down models stems from the need of computationally efficient credit models, that can be used in real-time. The model is used to pricing the credit derivative products, and to value the risk metrics of the products. Real-time requirement comes from the fact that the price of a product need to be quoted in a reasonably time to the counterparty before the market factors change significantly, and the risk numbers of a large portfolio of credit derivative products need to be calculated in a reasonably time in order to assess the open risks in a given market environment. The top-down approach simplifies the multi-name credit models into a single default counting process, which decreases the complexity of the model and decreases the computation times due to the smaller amount of calculations. The analogy can be found for example from stock indexes: when pricing an option on the stock index, the stock index is assumed to obey some stochastic process, which determines the behavior of the index. This is done on the index level, not on the individual stock level that belong into the index. This approach simplifies the option pricing model and makes the models computationally efficient.

The problem with the low model complexity is that the model is not able to address all the factors that affect the value of a credit derivative product. In order to assess the significance of this shortcoming, we test how well the model implied prices with the derived models in this thesis compare to the market prices on some credit derivative products, such as CDS index tranches. Also, we consider the model implied risk numbers, by calculating and plotting the model implied risk sensitivities of some credit derivative products, such as CDS index options.

Before moving into the modelling of credit derivative products with the generalized default counting process models, we introduce the pricing framework of CDS contracts. In Section 2, we begin by single-name CDS contracts, and continue with multi-name CDS contracts, such as CDS basket, index, and tranche contracts. The pricing of various credit derivative products are extensively covered in [9]. In Section 3, we introduce the general model of the default counting processes, which is the key in the modeling of CDS contracts in this thesis. In subsection 3.2, we rederive a model of intensity based default counting processes which is introduced in [6]. In Section 4, we consider methods to efficiently calibrate the default counting process model to market quotes. In Section 5, we price some credit derivative products with the default counting process model obtained in Section 3. In Section 6, we conclude the thesis.

2 Types of CDS contracts

The CDS contract consists of two parts, i.e. legs, which are called premium leg and protection leg. Premium leg consists of the regular payments protection buyer pays to the protection seller as a compensation for the transfer of the credit risk. The protection leg consists of the possible payments protection seller pays to protection buyer in case of credit event(s). Because the protection seller takes exposure for the credit risk, we say that protection seller is long credit risk, while protection buyer is hedging against the credit risk, and is therefore short credit risk.

In this section we introduce the pricing formulas for single-name and multi-name CDS contracts, that we use in the pricing models of the later sections. This section closely follows the formulations and methods discussed in [9]. We start from the standard single-name CDS contract and then move on to the multi-name contracts.

2.1 Single-name CDS contract

In the single-name CDS contract, the underlying is a single reference entity which typically is a company that has issued bonds. Thus the credit event, and the CDS protection payment, depends on whether the company defaults on the issued bonds. Let us take as an example the CDS contract of Nokia CDS EUR SR 5Y D14, which is a 5 year contract linked to Nokia Oyj. The maturity date of the contract is the end of the default protection period, if credit event has not occurred during the contract lifetime. Maturity dates are standardized to International Monetary Market (IMM) dates which are March 20th, June 20th, September 20th and December 20th. For example, if contract is traded on 24th of March 2019, the maturity date is June 20th 2024.

The premium leg consists of regular payments, which continue until the credit event or maturity date, whichever occurs sooner. If we denote the annual spread of premium payment from the trade date t_0 to maturity date t_N by $S(t_0, t_N)$, the expected discounted present value of the premium leg without accrued interest writes:

$$\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,single}}(t_0, t_N)\right] = S\left(t_0, t_N\right) \sum_{n=1}^{N} \Delta\left(t_{n-1}, t_n\right) Z\left(t_0, t_n\right) Q\left(t_0, t_n\right), \tag{2.1.1}$$

where N is the number of possible premium payments in the lifespan of the CDS contract from trade date to maturity date, $\Delta\left(t_{n-1}, t_n\right)$ is the day-count fraction of the year of the premium payment period $[t_{n-1}, t_n]$, $Z\left(t_0, t_n\right)$ is the discount factor from the time t_0 to t_n and $Q\left(t_0, t_n\right)$ is the probability that the reference entity will survive (i.e. avoid credit event) in the time period $[t_0, t_n]$. In the

Nokia CDS EUR SR 5Y D14 contract, the annualized spread $S(t_0, t_N)$ is 5% which is paid quarterly. The day-count convention of the contract is ACT/360 meaning that $\Delta(t_{n-1}, t_n)$ is the number of actual dates between t_n and t_{n-1} divided by 360.

If credit event occurs during the lifespan of the CDS contract, the accrued premium of the ongoing premium payment period is paid by the protection buyer in addition of the protection payment by the protection seller. To address the accrued premium, let us consider the probability that reference entity survives from time t_0 to time s and then after a small time period ds, credit event occurs. By denoting time of credit event by τ, this probability writes:

$$dP(s) = P(ds) = \mathbb{P}[(\tau < s+ds) \cap (\tau > s)] = \mathbb{P}[\tau < s+ds | \tau > s]\mathbb{P}[\tau > s] \quad (2.1.2)$$

The expected discounted value of accrued premium in a premium payment period $[t_{n-1}, t_n]$ is therefore:

$$\begin{aligned}\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Acc,single}}(t_{n-1}, t_n)\right] &= S(t_0, t_N) \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s) Z(t_0, s) dP(s) \\ &= S(t_0, t_N) \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s) Z(t_0, s)(-dQ(t_0, s)).\end{aligned} \quad (2.1.3)$$

The expression (2.1.3) can be approximated by the trapezoidal rule:

$$\begin{aligned}&\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Acc,single}}(t_{n-1}, t_n)\right] \\ &\approx \frac{S(t_0, t_N)}{2}\left(\underbrace{\Delta(t_{n-1}, t_{n-1})}_{=0} Z(t_0, t_{n-1}) + \Delta(t_{n-1}, t_n) Z(t_0, t_n)\right)(Q(t_0, t_{n-1}) - Q(t_0, t_n)) \\ &= \frac{S(t_0, t_N)}{2} \Delta(t_{n-1}, t_n) Z(t_0, t_n)(Q(t_0, t_{n-1}) - Q(t_0, t_n)).\end{aligned} \quad (2.1.4)$$

By combining the expressions (2.1.1) and (2.1.4), we get the value of premium leg with accrued interest:

$$\begin{aligned}&\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{CDS,single}}(t_0, t_N)\right] \\ &\approx \frac{S(t_0, t_N)}{2} \sum_{n=1}^{N} \Delta(t_{n-1}, t_n) Z(t_0, t_n)(Q(t_0, t_{n-1}) + Q(t_0, t_n)).\end{aligned} \quad (2.1.5)$$

The expression (2.1.5) excluding the coupon S is often called the RPV01, i.e. the risky present value unit. This means:

$$RPV01(t_0, t_N) = \frac{1}{S\left(t_0, t_N\right)} \mathbb{E}\left[\mathrm{PV}^{(\mathrm{premium})}_{\mathrm{CDS,single}}(t_0, t_N)\right]. \tag{2.1.6}$$

It is important to notice that the expression (2.1.5) assumes the trade date t_0 to fall on one of the premium payment dates of the contract, i.e. it omits the accrued premium during the ongoing premium payment period up to trade date t_0. We stick to this assumption due to simplicity.

In case of credit event, protection leg pays protection buyer payment of $(1 - R)$ per currency unit of the notional of the CDS contract, where R denotes the recovery rate of the underlying reference entity. To model the expected protection leg payment, we use the same approach as in addressing the accrued premium in premium leg. By considering the probability that reference entity survives from time t_0 to time s and then after a small time period ds, credit event occurs. The expected discounted protection payment is:

$$\begin{aligned}
\mathbb{E}\left[\mathrm{PV}^{(\mathrm{protection})}_{\mathrm{CDS,single}}\left(t_0, t_N\right)\right] &= (1 - R) \int_{t_0}^{t_N} Z(t_0, s) dP(s) \\
&= (1 - R) \int_{t_0}^{t_N} Z(t_0, s)(-dQ(t_0, s)).
\end{aligned} \tag{2.1.7}$$

By using M interval points per year to approximate the integral in the expression (2.1.7), the trapezoidal rule gives:

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{PV}^{(\mathrm{protection})}_{\mathrm{CDS,single}}\left(t_0, t_N\right)\right] \\
&\approx \frac{(1 - R)}{2} \sum_{m=1}^{M * t_N} \left(Z\left(t_0, t_{m-1}\right) + Z\left(t_0, t_m\right)\right)\left(Q\left(t_0, t_{m-1}\right) - Q\left(t_0, t_m\right)\right).
\end{aligned} \tag{2.1.8}$$

As we can see from the expressions (2.1.5) and (2.1.8), i.e. premium leg and protection leg, respectively, the only parameter that is not directly observable is the survival probability function $Q(t_0, t)$. The pricing models of the single-name CDS contracts differ in how they model this survival probability function. We consider the modeling of $Q(t_0, t)$ in the Section 3.

2.2 CDS index contract

As mentioned in the Section 1, CDS index contracts have become one of the most popular credit products since the inception in the beginning of 2000's. The main CDS indices are under the CDX and iTraxx index families, which are governed by Markit. The indices are grouped geographically, and include North American, European, Japanese, Asian, Australian and emerging market indices.

The CDX and iTraxx indices roll semiannually on the IMM dates described in Section 2.1. The rolling of the index means that a new series of the index is introduced. The underlying firm list of the index is updated in every roll so that the company list obeys the index rules. The list of firms belonging to a specific index is determined by so-called liquidity list which is provided by Markit.

The valuation of CDS index legs is analogous to pricing the legs of single-name CDS. Let us assume that there is m underlying firms in the CDS contract, and that firms have weights $\{w_k\}_{k=1}^m$ in the index, where $\sum_{k=1}^m w_k = 1$. Equal weighting would be $w_k = \frac{1}{m}, \forall k$. Let us denote the set of survival probability functions by $\{Q_k(t_0,t)\}_{k=1}^m$.

The value of the premium leg can be written according to expression (2.1.5):

$$\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{CDS,index}}(t_0,t_N)\right] = \mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_0,t_N)\right] + \mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Acc,index}}(t_0,t_N)\right], \tag{2.2.1}$$

where interest payment part and accrued interest part can be written according to (2.1.1) and (2.1.3), respectively:

$$\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_0,t_N)\right] = S\left(t_0,t_N\right)\sum_{n=1}^{N}\sum_{k=1}^{m} w_k \Delta\left(t_{n-1},t_n\right) Z\left(t_0,t_n\right) Q_k\left(t_0,t_n\right), \tag{2.2.2}$$

$$\begin{aligned}\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Acc,index}}\left(t_0,t_N\right)\right] &= S\left(t_0,t_N\right)\sum_{n=1}^{N}\sum_{k=1}^{m} w_k \int_{t_{n-1}}^{t_n} \Delta\left(t_{n-1},s\right) Z(t_0,s) dP_k(s) \\ &= S\left(t_0,t_N\right)\sum_{n=1}^{N}\sum_{k=1}^{m} w_k \int_{t_{n-1}}^{t_n} \Delta\left(t_{n-1},s\right) Z(t_0,s)\left(-dQ_k(t_0,s)\right),\end{aligned} \tag{2.2.3}$$

where $P(s)$ is defined in (2.1.2). The expression (2.2.1) is often called intrinsic value of the premium leg since it considers the individual survival probabilities $\{Q_k(t_0,t)\}_{k=1}^m$ instead of modeling only the survival probability of the whole index.

The protection leg is written according to expression (2.1.7):

$$
\begin{aligned}
\mathbb{E}\left[\mathrm{PV}^{(\text{protection})}_{\text{CDS,index}}\left(t_0, t_N\right)\right] &= \sum_{k=1}^{m} w_k\left(1-R_k\right)\int_{t_0}^{t_N} Z(t_0, s)dP_k(s) \\
&= \sum_{k=1}^{m} w_k\left(1-R_k\right)\int_{t_0}^{t_N} Z(t_0, s)\left(-dQ_k(t_0, s)\right),
\end{aligned}
\tag{2.2.4}
$$

where we assume individual recovery rates $\{R_k\}_{k=1}^{m}$.

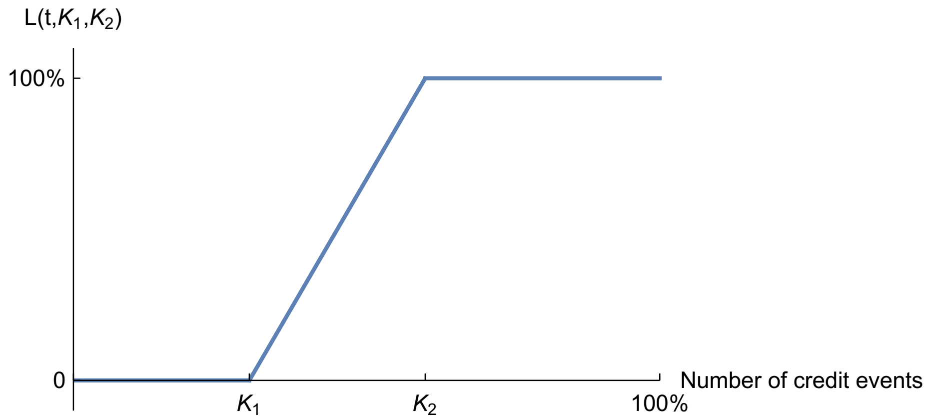

Figure 2: The loss process $L^{(tranche)}(t, K_1, K_2)$ as a function of credit events

2.3 Tranched index contract

In addition to taking a position on the whole credit index, many credit indices enables investor to take a position on a part of a credit index called tranche. Tranche is defined in terms of the attachment point and detachment point, denoted in percentages. These points denote the interval of credit events in the index in which the protection payment is delivered to the protection buyer by the protection seller. In the Markit CDX and iTraxx credit indices, there are standardized tranches that are quoted in the market. The standard tranches on CDX and iTraxx indices are quoted on the every second series of the credit indices to provide adequate liquidity.

In Figure 2 the loss process $L^{(tranche)}(t, K_1, K_2)$ is shown at time instant t with attachment and detachment points K_1 and K_2, respectively. The loss process is the percentage amount of credit events in the protection position, and can be written as:

$$L^{(tranche)}\left(t, K_1, K_2\right) = \frac{\max\left\{L^{(index)}(t) - K_1, 0\right\} - \max\left\{L^{(index)}(t) - K_2, 0\right\}}{K_2 - K_1}, \tag{2.3.1}$$

where $L^{(index)}(t)$ denotes the loss process of the whole index. The survival process $Q(t_0, t)$ can be written in terms of the tranche and index loss processes as follows:

$$\begin{aligned} Q^{(tranche)}(t_0, t) &= 1 - \mathbb{E}\left[L^{(tranche)}\left(t, K_1, K_2\right)\right] \\ &= 1 - \frac{\max\left\{\mathbb{E}[L^{(index)}(t)] - K_1, 0\right\} - \max\left\{\mathbb{E}[L^{(index)}(t)] - K_2, 0\right\}}{K_2 - K_1}. \end{aligned} \tag{2.3.2}$$

CDS index tranches are quoted by an upfront payment and spread, depending on the seniority of the tranche. For example, in the Table 1 we see market quotes on Markit iTraxx Europe MAIN s30 5y index tranches:

Table 1: Markit iTraxx Europe MAIN s30 5y (20-Dec-2023) Ref 62				
Tranche	Bid	Ask	Delta	Quote
0-3%	39.0625	39.8125	9.2	percentage point upfront + 1%
3-6%	8.3750	8.8750	5.2	percentage point upfront + 1%
6-12%	115.5000	120.5000	2.5	Spread, 0% recovery
12-100%	20.3750	21.6250	0.48	Spread, 40% recovery

In the Table 1, the price on equity and junior tranches 0-3% and 3-6%, respectively, is quoted in the market as a basis point upfront, denoted by F, plus fixed annualized premium (i.e., spread) of 1%. For mezzanine and senior tranches 6-12% and 12-100%, respectively, the price is quoted in annualized premium in basis points assuming a given recovery value for credit events. The delta is defined as $\frac{\text{Tranche DV01}}{\text{Index DV01}}$, where DV01 is the difference in value (i.e., DV) when the spread/upfront payment increases by one basis point (i.e., 01). It is important to notice that delta is an decreasing function of the tranche seniority, i.e., the value of a subordinated tranche is more sensitive to change in spread than the value of more senior tranche. Also, taking a long position in all of the tranches with the weights of $K_2 - K_1$ is equivalent to taking a long position in the whole index, meaning that the delta of the portfolio of the combined tranche positions equals to one. By using the market quotes in Table 1, the portfolio delta is $0.03*9+0.03*5.2+0.06*2.5+0.88*0.48 = 0.9984$. The Ref 62 denotes the market quote of flat spread of the index which is 62 basis points, or 0.62%.

The protection leg of tranched index contract writes similarly to the index protection leg in expression (2.2.4) with weights $w_k = 1/m$ and constant recovery rate R:

$$\begin{aligned}\mathbb{E}\left[\mathrm{PV}^{(\text{protection})}_{\text{CDS,tranche}}(t_0,t_N)\right] &= (1-R)\int_{t_0}^{t_N} Z(t_0,s)\left(-dQ^{(tranche)}(t_0,s)\right)\\ &\overset{(2.3.2)}{=} (1-R)\int_{t_0}^{t_N} Z(t_0,s)\mathbb{E}\left[dL^{(tranche)}(t,K_1,K_2)\right]. \qquad (2.3.3)\end{aligned}$$

The premium leg consists of the upfront payment F that is a percentage value of the notional of the contract, and regular premium payments. By excluding the accrued interest, the premium leg writes similar to expression (2.2.2):

$$\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,tranche}}(t_0,t_N)\right] = F + S\left(t_0,t_N\right)\sum_{n=1}^{N}\Delta\left(t_{n-1},t_n\right)Z\left(t_0,t_n\right)Q^{(tranche)}\left(t_0,t_n\right)$$

$$\overset{(2.3.2)}{=} F + S\left(t_0,t_N\right)\sum_{n=1}^{N}\Delta\left(t_{n-1},t_n\right)Z\left(t_0,t_n\right)\mathbb{E}\left[1 - L^{(tranche)}\left(t_i,K_1,K_2\right)\right]. \tag{2.3.4}$$

2.4 CDS index options

CDS index options are bilateral agreements in which one counterparty, i.e., option holder, has a right, but not the obligation, to enter into a CDS index contract on prespecified date(s) on agreed premium coupon K, known as the strike of the option. The option type can be either payer, in which option holder can enter into long protection position (paying premium leg and receiving protection leg of the CDS contract), or receiver, in which option holder can enter into short protection position.

To simplify the analysis, let us constraint our consideration to European style option contracts. In European CDS index options, we have three important dates in determining the value of the contract, which are option initiation date t_0, option expiry date t_E and CDS index maturity date T, where $t_E < T$. The payoff of the option consists of three parts:

1. The settlement of the front-end protection period, denoted by FEP. The front-end protection period is between t_0 and t_E, and consists of the delivery of protection payments from protection seller to protection buyer at the expiry date t_E. By assuming index of m constituents with default times $\{\tau^k\}_{k=1}^m$, the value of front-end protection payment FEP writes:

$$FEP = \sum_{k=1}^{m}\mathbb{E}\left[\mathbf{1}_{\{\tau^k \le t_E\}}(1-R^k)\right], \tag{2.4.1}$$

   where R^k is the recovery rate of the k'th constituent.

2. The exercise price of the option, denoted by $G(K)$. The exercise price can be written in case of the payer option with strike K and CDS index coupon $S(t_E,T)$ as follows:

$$G(K) = \frac{K - S(t_E,T)}{S(t_E,T)}\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_E,T)\right], \tag{2.4.2}$$

   where $\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_E,t_N)\right]$ is given in expression (2.2.2). In case of receiver option, we simply put minus sign in front of the expression (2.4.2). The exercise price is paid at exercise date t_E if option is exercised.

3. The value of the underlying CDS contract on exercise date, denoted by VC. The value of the CDS contract is simply the difference between the value of the protection leg and the premium leg. In case of the receiver option, we have:

$$VC(t_E, T) = \mathbb{E}\left[\mathrm{PV}^{(\text{protection})}_{\text{CDS,index}}(t_E, T)\right] - \mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_E, T)\right], \tag{2.4.3}$$

where expected present values are given in expressions (2.2.2) and (2.2.4).

Denote $H(t_E) = FEP + VC(t_E, T)$. Then the intrinsic values of the payer and receiver options at exercise date t_E, denoted by $V^{payer}(t_E)$ and $V^{receiver}(t_E)$, are:

$$V^{payer}(t_E) = \max\{H(t_E) - G(K), 0\}, \tag{2.4.4}$$
$$V^{receiver}(t_E) = \max\{G(K) - H(t_E), 0\}. \tag{2.4.5}$$

The value of the payer option at the contract initiation date t_0 is given as:

$$V^{payer}(t_0) = \mathbb{E}\left[e^{-\int_{t_0}^{t_E} r(s)ds} \max\{H(t_E) - G(K), 0\}\right] \tag{2.4.6}$$

where $r(s)$ is the short rate at time t. The value of the receiver option at t_0 is obtained by replacing $V^{payer}(t_E)$ by $V^{receiver}(t_E)$.

3 Default counting processes

As we saw in the Section 2, the valuation of CDS contracts, and credit derivatives in general, boil down to determining the survival probability of the reference entity. In Section 2, the survival probability was denoted by $Q(t_0, t)$ and writes:

$$Q(t_0, t) = \mathbb{P}[\tau > t | \tau > t_0] = \mathbb{E}[\mathbb{1}_{\{\tau > t\}} | \tau > t_0], \tag{3.0.1}$$

where τ is the time when the reference entity faces a credit event and $\mathbb{1}_{\{\tau > t\}} = \begin{cases} 1, & \tau > t \\ 0, & \tau \leq t \end{cases}$ is an indicator function. From the expression (3.0.1), we see that the survival probability can be written with a default counting process as:

$$\mathbb{E}[N(t) | N(t_0) = 0] = 1 - Q(t_0, t), \tag{3.0.2}$$

where $N(t) = \mathbb{1}_{\{\tau \leq t\}}$. Thus it suffices to just consider the default counting process defined by expression (3.0.2).

To generalize the analysis, let us assume that there exists a filtered probability space $\left(\Omega, \{\mathcal{F}_t\}_{t \in \mathbb{R}_+}, \mathbb{P}\right)$ for the state of the world at time t. The filtration $\mathcal{F}$ is assumed to be right-continuous, meaning that $\mathcal{F}_t = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$, where ϵ is infinitely small increase to time t. We have set of m random variables $\tau = \left\{\tau^k\right\}_{k=1}^m$, where $\tau^k > 0$ denotes the time of credit event of the k'th firm. The conditional expected value (3.0.2) for the k'th firm becomes $\mathbb{E}[N^k(t) | \mathcal{F}_{t_0}]$.

Let us introduce an ordered sequence of realized times of credit events $T = \left\{T^k\right\}_{k=1}^m$. We require that sequence T is strictly increasing, which implies that more than one credit event cannot happen in exactly the same time. Also by requiring that $T^m \xrightarrow{m \to \infty} \infty$, i.e. an infinite amount of credit events can only happen in infinite time, we have that the counting processes $N^k(t) = \mathbb{1}_{\{\tau^k \leq t\}}$ and $O^k(t) = \mathbb{1}_{\{T^k \leq t\}}$ are locally integrable semimartingales (see thorough discussion in [10]). Therefore, the default counting process:

$$N(t) = \sum_{k=1}^{m} N^k(t) = \sum_{k=1}^{m} O^k(t) \tag{3.0.3}$$

is semimartingale because the sum of semimartingales is a semimartingale. The counting processes $N = \{N(t)\}_{t \in \mathbb{R}_+}$, $N^k = \left\{N^k(t)\right\}_{t \in \mathbb{R}_+}$ and $O^k = \left\{O^k(t)\right\}_{t \in \mathbb{R}_+}$ are right continuous with jump size of 1. Also the value of the counting process in arbitrary time interval $(s, t]$ can be written as a difference of counting processes:

$$N((s, t]) = N(t) - N(s). \tag{3.0.4}$$

By the definition of semimartingale, default counting processes satisfying aforementioned properties can be decomposed into martingale $M = \{M(t)\}_{t\in\mathbb{R}_+}$ and finite variation process $A = \{A(t)\}_{t\in\mathbb{R}_+}$ according to Doob-Meyer decomposition:

$$N(t) = M(t) + A(t), \tag{3.0.5}$$

where $M(t_0) = 0$ and $A(t_0) = 0$, because we assume $N(t_0) = 0$. Also for individual firm counting processes we have $N^k(t_0) = M^k(t_0) + A^k(t_0)$ and $M^k(t_0) = 0$, $A^k(t_0) = 0$.

Because martingales have property $\mathbb{E}\left[M(t)\,|\mathcal{F}_s\right] = M(s), \forall t > s$, we have:

$$\begin{aligned}
\mathbb{E}\left[N(t)\,|\mathcal{F}_s\right] &= \mathbb{E}\left[M(t) + A(t)\,|\mathcal{F}_s\right] \\
&= \underbrace{\mathbb{E}\left[M(t)\,|\mathcal{F}_s\right]}_{=M(s)} + \mathbb{E}\left[A(t)\,|\mathcal{F}_s\right] \\
&= M(s) + \mathbb{E}\left[A(t)\,|\mathcal{F}_s\right].
\end{aligned} \tag{3.0.6}$$

Based on the expression (3.0.6), the expected value of (3.0.4) can be written as:

$$\begin{aligned}
\mathbb{E}[N((s,t])\,|\mathcal{F}_s] &= \mathbb{E}\left[N(t) - N(s)\,|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[N(t)\,|\mathcal{F}_s\right] - \mathbb{E}\left[N(s)\,|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[A(t)\,|\mathcal{F}_s\right] - \mathbb{E}\left[A(s)\,|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[A(t) - A(s)\,|\mathcal{F}_s\right].
\end{aligned} \tag{3.0.7}$$

Expression (3.0.7) tell us that the expected number of credit events in time interval $(s,t]$ is completely determined by the compensator of the counting process. If we partition the time interval $(s,t]$ to time intervals $\{[s_k, s_{k-1}]\}_{k=1}^K$, where $s_0 = s$ and $s_K = t$, and denote interval length by $\epsilon = s_k - s_{k-1}$, we can write a sum:

$$\begin{aligned}
A(t) &=_{a.s.} \lim_{\epsilon\to 0} \sum_{k=1}^{K} \mathbb{E}\left[A(s_k) - A(s_k - \epsilon)|\mathcal{F}_{s_k-\epsilon}\right] \\
&\overset{(3.0.7)}{=} \lim_{\epsilon\to 0} \sum_{k=1}^{K} \mathbb{E}\left[N(s_k) - N(s_k - \epsilon)|\mathcal{F}_{s_k-\epsilon}\right].
\end{aligned} \tag{3.0.8}$$

In expression (3.0.8) the almost sure equivalence comes from the fact that

$\mathbb{P}\left[A(s_k) = \mathbb{E}\left[A(s_k)|\mathcal{F}_{s_k-\epsilon}\right]\right] \to 1$ as $\epsilon \to 0$. Summation (3.0.8) can equivalently be written as an integral:

$$A(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_0}^{t} \mathbb{E}\left[N(s) - N(s-\epsilon)\,|\mathcal{F}_{s-\epsilon}\right] ds. \tag{3.0.9}$$

Because counting process N is non-decreasing, the finite variation process A is also non-decreasing. We can rewrite the expression (3.0.9) as:

$$A(t) = \int_{t_0}^{t} \lambda(s) ds, \tag{3.0.10}$$

where $\lambda(t) \geq 0$ and possibly discontinuous function.

As an example, consider a standard Poisson process with probability distribution function for counting process increments, denoted by X, in time interval ϵ as $f(x) = \frac{(\nu\epsilon)^x e^{-\nu\epsilon}}{x!}$, where x is non-negative integer. By recalling that $e^a = \sum_{b=0}^{\infty} \frac{a^b}{b!}$, the expected value for the increments can be calculated to be $\mathbb{E}[X] = \sum_{x=0}^{\infty} x f(x) = \sum_{x=0}^{\infty} x \frac{(\nu\epsilon)^x e^{-\nu\epsilon}}{x!} = \nu\epsilon e^{-\nu\epsilon} e^{\nu\epsilon} = \nu\epsilon$. Thus, in expression (3.0.9) we have $\mathbb{E}\left[N(s) - N(s-\epsilon)\,|\mathcal{F}_{s-\epsilon}\right] = \nu\epsilon$, which implies $A(t) = \nu(t - t_0)$, and $\lambda(t) = \nu$ in expression (3.0.10).

The finite variation processes A and A^k are called compensators (or $\mathcal{F}$-compensators) of the associated semimartingale processes. The compensators A and A^k have continuous paths if and only if $\tau^k, i \in \{1, ..., n\}$, are totally inaccessible, meaning that $\mathbb{P}\left[\tau^k = t\right] = 0$. This implies that the counting process N is almost surely (a.s., i.e., with probability one) continuous at stopping times τ^k, so $\mathbb{P}\left[X_\tau = X_{\tau-}\right] = 1$ ($\tau-$ denoting approach of τ as $\lim_{\epsilon \to 0, \epsilon > 0} \tau - \epsilon$), which can be rewritten as $\mathbb{P}\left[X_\tau - X_{\tau-} = 0\right] = 1$. This property is called quasi-left-continuity of counting process N (again, thorough discussion can be found from [10]). The totally inaccessible condition implies that the integral in (3.0.10) is well defined. In general, if the compensator can be expressed as (3.0.10), the process N is called intensity based.

In case of quasi-left-continuous strictly increasing counting process N with compensator A, we know that A is continuous strictly increasing process such that

$$\exp\left(aN(t) - \left(e^a - 1\right) A(t)\right) \text{ and } N(t) - A(t) \tag{3.0.11}$$

are local martingales (denoted by $\mathcal{M}_{c,\text{loc}}$) for all $a \in \mathbb{C}$, where $\mathbb{C}$ is a set of complex numbers. This result can be used to check that A is a compensator of N.

Observe next that based on the expressions (3.0.3) and (3.0.5), we can write the compensator and martingale process of the default counting process N in terms of the constituent processes as:

$$
\begin{aligned}
N(t) &= \sum_{k=1}^{m} N^k(t) \\
\Leftrightarrow M(t) + A(t) &= \sum_{k=1}^{m} \left(M^k(t) + A^k(t) \right) \\
\Leftrightarrow M(t) + A(t) &= \left[\sum_{k=1}^{m} M^k(t) \right] + \left[\sum_{k=1}^{m} A^k(t) \right] \\
\Rightarrow M(t) &= \sum_{k=1}^{m} M^k(t) \text{ and } A(t) = \sum_{k=1}^{m} A^k(t)
\end{aligned} \tag{3.0.12}
$$

According to the expression (3.0.12), the compensator A for counting process N can be constructed as a sum of compensators A^k for indicator processes N^k. If processes N^k are intensity based, i.e. $A^k(t) = \int_{t_0}^{t} \lambda^k(s)\, ds$, we have:

$$
\lambda(t) = \sum_{k=1}^{m} \lambda^k(t) \tag{3.0.13}
$$

The approach in the expressions (3.0.12) and (3.0.13) is called bottom up - approach, i.e., we know compensators of the individual firms, and with these we construct the compensator of the bundle of firms (or economy in general). The opposite approach is top down -approach in which compensator of the bundle of firms is given, and we aim to construct the compensators for individual firms. The partitioning of the bundle compensator into the individual firm compensators is done via random thinning. We consider random thinning in the next subsection.

3.1 Random Thinning of default counting processes

In the class of top-down models, we model the aggregate counting process N by determining the dynamics for the compensator A. In the intensity based models, this is done by choosing a non-negative intensity function λ. The choice of λ uniquely determines the expected behavior of N. However, as λ only defines the aggregate compensator, it does not say anything in particular on what is the expected behavior of the constituent firms. In order to determine the expected behavior of the constituent firms based on the dynamics of aggregate counting process, we need to introduce a point process operation called random thinning.

Random thinning is an operation that thins the aggregate point process, i.e., removes some points from the aggregate point process, according to some specified rule that may be deterministic or stochastic. The thinned process is another point process that is generated by a subset of constituents. When applied to aggregate default counting process, the thinned process is a default counting process on a subset of firms, or on a single firm, in the aggregate list of firms. Point process thinning is extensively covered in [11].

In order to thin the aggregate counting process, we need to specify a process, either stochastic or deterministic, that determines how aggregate process is thinned. Let us introduce process $Z^k(t)$ that is bounded, non-negative, $\mathcal{F}$-predictable (i.e. $Z^k(t) \in m\mathcal{F}_{t-\epsilon}$, where $m\mathcal{F}_{t-\epsilon}$ denotes that inverse image $(Z^k(t))^{-1} \in \mathcal{F}_{t-\epsilon}$), and satisfies $\int_{t_0}^t Z^k(s)dN(s) = N^k(t)$, where N^k is the counting process of the k'th constituent firm, almost surely. Based on these assumptions and expression (3.0.5), we can write N^k as:

$$
\begin{aligned}
N^k(t) &= \int_{t_0}^t Z^k(s)dN(s) \\
&\overset{(3.0.5)}{=} \int_{t_0}^t Z^k(s)d(A(s) + M(s)) \\
&= \int_{t_0}^t Z^k(s)dA(s) + \int_{t_0}^t Z^k(s)dM(s).
\end{aligned}
\tag{3.1.1}
$$

Our aim is to construct the compensator of the counting process $N^k(t)$ based on the integral expression (3.1.1). To do this, let us define the stochastic integration as a limit of sum:

$$
(X \bullet Y)_t^n = \sum_{k=1}^{\lfloor 2^n t \rfloor} X_{(k-1)2^{-n}} \left(Y_{k2^{-n}} - Y_{(k-1)2^{-n}} \right), \tag{3.1.2}
$$

where X and Y are two random processes and $\lfloor \bullet \rfloor$ is the floor operation. Clearly, we have $(X \bullet Y)_t^n \overset{n\to\infty}{\longrightarrow} \int_{t_0}^t XdY$, where $X_0 = X_{t_0}$ and $Y_0 = Y_{t_0}$. With this definition, we can write expected value:

$$
\begin{aligned}
&\mathbb{E}\left[\int_{t_0}^{t} Z^k(s)dM(s)|\mathcal{F}_{t_0}\right] = \mathbb{E}\left[\lim_{n\to\infty}\sum_{k=1}^{\lfloor 2^n t\rfloor} Z_{(k-1)2^{-n}}\left(M_{k2^{-n}} - M_{(k-1)2^{-n}}\right)|\mathcal{F}_{t_0}\right]\\
&= \lim_{n\to\infty}\sum_{k=1}^{\lfloor 2^n t\rfloor} \mathbb{E}\left[Z_{(k-1)2^{-n}}\left(M_{k2^{-n}} - M_{(k-1)2^{-n}}\right)|\mathcal{F}_{t_0}\right]\\
&\overset{*}{=} \lim_{n\to\infty}\sum_{k=1}^{\lfloor 2^n t\rfloor} \mathbb{E}\left[Z_{(k-1)2^{-n}}\mathbb{E}\left[M_{k2^{-n}} - M_{(k-1)2^{-n}}|\mathcal{F}_{(k-1)2^{-n}}\right]|\mathcal{F}_{t_0}\right]\\
&= \lim_{n\to\infty}\sum_{k=1}^{\lfloor 2^n t\rfloor} \mathbb{E}\left[Z_{(k-1)2^{-n}}\left(\underbrace{\mathbb{E}\left[M_{k2^{-n}}|\mathcal{F}_{(k-1)2^{-n}}\right]}_{=M_{(k-1)2^{-n}}} - \underbrace{\mathbb{E}\left[M_{(k-1)2^{-n}}|\mathcal{F}_{(k-1)2^{-n}}\right]}_{=M_{(k-1)2^{-n}}}\right)|\mathcal{F}_{t_0}\right]\\
&= \int_{t_0}^{t_0} Z^k(s)dM(s). \qquad (3.1.3)
\end{aligned}
$$

(*) *We need to use two basic properties of the conditional expectation. The first property states that if $X \in m\mathcal{F}$, then $\mathbb{E}[XY|\mathcal{F}] = X\mathbb{E}[Y|\mathcal{F}]$. The second property states that with filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t=t_0}^{T}$, we have $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[X|\mathcal{F}_s]$, where $s < t$, since $\mathcal{F}_s \subseteq \mathcal{F}_t$ (so called law of total expectation). According to our assumptions, $Z^k(t) \in m\mathcal{F}_t$.*

Based on expression (3.1.3), integral $\int_{t_0}^{t} Z^k(s)dM(s)$ is martingale. Therefore, the compensator of the default counting process $N^k(t)$ in the expression (3.1.1) writes:

$$
A^k(t) = \int_{t_0}^{t} Z^k(s)dA(s), \qquad (3.1.4)
$$

where A is compensator to process N. By using expressions (3.0.10) and (3.1.4), we can write the relationship between the index intensity process λ and the k'th constituent intensity process λ^k as:

$$
\begin{aligned}
&\int_{t_0}^{t} \lambda^k(s)\,ds = \int_{t_0}^{t} Z^k(s)\,dA(s)\\
&\Rightarrow \lambda^k(s)ds = Z^k(s)dA(s)\\
&\Rightarrow A(t) - \underbrace{A(t_0)}_{=0} = \int_{t_0}^{t} \frac{\lambda^k(s)}{Z^k(s)}\,ds\\
&\overset{(3.0.10)}{\Rightarrow} \int_{t_0}^{t} \lambda(s)\,ds = \int_{t_0}^{t} \frac{\lambda^k(s)}{Z^k(s)}\,ds\\
&\Rightarrow \lambda^k(t) = Z^k(t)\lambda(t) \qquad (3.1.5)
\end{aligned}
$$

By noticing that the sum of constituent intensity processes $\{\lambda^k\}_{k=1}^m$ equal to the index intensity process λ, as noted in expression (3.0.13), we obtain one additional property of Z^k:

$$\underbrace{\sum_{k=1}^{m}\lambda^k(t)}_{=\lambda(t)} = \lambda(t)\sum_{k=1}^{m} Z^k(t) \Rightarrow \sum_{k=1}^{m} Z^k(t) = 1. \tag{3.1.6}$$

Finally, we need to set $Z^k(t) = 0$, when the k'th firm has defaulted in order to address the fact that defaulted constituent is not adding points to the index counting process after default. After this observation we are ready to collect the properties that $Z^k(t), \forall k \in \{1, 2, ..., m\}$, need to satisfy:

$$\begin{cases} Z^k \geq 0, & (3.1.7a) \\ \sum_{k=1}^{m} Z^k(t) = 1 \text{ when } t \leq T^m, & (3.1.7b) \\ Z^k = 0 \text{ when } t > \tau^k. & (3.1.7c) \end{cases}$$

Notice that the properties (3.1.7a)-(3.1.7c) are properties that probabilities satisfy. This is not a coincidence as the processes $\{Z^k\}_{k=1}^m$ is defined based on probabilities that we consider in detail later in this subsection.

For computational purposes it is important to consider a case in which we can identify the default times $T^k, k \in \{1, ..., m\}$ but not the identity of the defaulter. We can model such system by introducing a sub-filtration $\left\{\bar{\mathcal{F}}_t\right\}_{t\in\mathbb{R}_+}$ such that $\bar{\mathcal{F}}_t \subset \mathcal{F}_t$. The intensity parameter $\bar{\lambda}^k(t)$ in such system is the expected value of intensity parameter $\lambda^k(t)$ given information at time t by the sub-filtration:

$$\bar{\lambda}^k(t) = \mathbb{E}\left[\lambda^k(t) \,\middle|\, \bar{\mathcal{F}}_t\right]. \tag{3.1.8}$$

From (3.1.8) it is important to notice that $\bar{\lambda}^k(t)$ does not vanish until $t \leq T^m$ because we cannot identify the identity of the defaulter, only default times. This means that, when introducing thinning process $\bar{Z}^k(t)$, thinning process does not vanish until $t \leq T^m$, according to the expression (3.1.8). Thus the thinning process $\bar{Z}^k(t)$ satisfy properties:

$$\begin{cases} \bar{Z}^k \geq 0, & (3.1.9a) \\ \sum_{k=1}^{m} \bar{Z}^k(t) = 1 \text{ when } t \leq T^m, & (3.1.9b) \\ \bar{Z}^k = 0 \text{ when } t > T^m. & (3.1.9c) \end{cases}$$

The difference between Z^k and $\bar{Z}^k$ is characterized by properties (3.1.7c) and (3.1.9c). In (3.1.9c) the thinning process is zero almost surely only if all firms in the system have defaulted while in (3.1.7c) the thinning process is almost surely zero only if the k'th firm has defaulted. The extension of Z^k to $\bar{Z}^k$ is called smoothing of the thinning process. In the subsection 3.2. we see the importance of smoothing for the computational purposes.

In [6], processes $Z^k(t)$ and $\bar{Z}^k(t)$ are written in the form:

$$Z^k(t) = \sum_{j=1}^{m} \mathbb{P}\left[\tau^k = T^j | \mathcal{F}_{t-}\right] \mathbb{1}_{\{T^{j-1} < t \leq T^j\}}, \tag{3.1.10}$$

$$\bar{Z}^k(t) = \sum_{j=1}^{m} \mathbb{P}\left[\tau^k = T^j | \bar{\mathcal{F}}_{t-}\right] \mathbb{1}_{\{T^{j-1} < t \leq T^j\}}, \tag{3.1.11}$$

and it is straightforward to check that (3.1.10) and (3.1.11) satisfy the properties (3.1.7a)-(3.1.7c) and (3.1.9a)-(3.1.9c), respectively.

By (3.1.10) and (3.1.11), the interpretation of $Z^k(t)$ and $\bar{Z}^k(t)$ is that they are the market's view at time t of the probability that the k'th firm is the next firm to default in a state given by the information $\mathcal{F}_t$ and $\bar{\mathcal{F}}_t$, respectively. The expressions (3.1.10) and (3.1.11) can be equivalently written as:

$$Z^k(t) = \begin{cases} \lim_{\epsilon \to 0} \frac{\mathbb{P}\left(t < \tau^k \leq t+\epsilon | \mathcal{F}_t\right)}{\mathbb{E}(N(t+\epsilon) - N(t) | \mathcal{F}_t)}, & \lim_{\epsilon \to 0} \frac{\mathbb{P}\left(t < \tau^k \leq t+\epsilon | \mathcal{F}_t\right)}{\mathbb{E}(N(t+\epsilon) - N(t) | \mathcal{F}_t)} < \infty \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.12}$$

$$\bar{Z}^k(t) = \begin{cases} \lim_{\epsilon \to 0} \frac{\mathbb{P}\left(t < \tau^k \leq t+\epsilon | \bar{\mathcal{F}}_t\right)}{\mathbb{E}\left(N(t+\epsilon) - N(t) | \bar{\mathcal{F}}_t\right)}, & \lim_{\epsilon \to 0} \frac{\mathbb{P}\left(t < \tau^k \leq t+\epsilon | \bar{\mathcal{F}}_t\right)}{\mathbb{E}\left(N(t+\epsilon) - N(t) | \bar{\mathcal{F}}_t\right)} < \infty \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.13}$$

In the next subsection we consider intensity based default counting processes in more detail. We introduce specific diffusion processes for the intensity parameter $\lambda(s)$, and derive a computationally efficient method to solve survival and default probabilities induced by the given diffusion process.

3.2 Intensity based default counting processes

As we discussed in Section 3, very mild assumptions on the behavior of the default counting process N enabled us to model N as a semimartingale, i.e., sum of finite variation process A, called compensator, and martingale process M. One of the assumptions was to bound the jumps of N to one, which essentially means that only one credit event can happen at a given time instance. This essentially means that N is locally integrable, i.e., the difference $\Delta N = N_t - N_s, s < t$, is bounded. The local integrability of N ensures that the compensator exists. Moreover, the compensator can be written in terms of non-negative function λ, called intensity process.

In this subsection, we show that the choice of intensity process completely characterizes the expected behavior of N, and everything we need to know about default counting process to price credit derivatives. Therefore, the pricing model is completely characterized by the choice of the intensity process. We follow the derivation of intensity based default counting processes introduced in [6] closely and derive all the intermediate steps with the results obtained already in Section 3.

Let us consider a system of m firms with individual firm credit event indicator processes $N_t^k = N^k(t) = \mathbb{1}_{\{\tau^k \leq t\}}, k \in \{1, ..., m\}$. The index default counting process is $N_t = \sum_{k=1}^m N_t^k$, according to (3.0.3). The compensator A_t of N_t has following form according to (3.0.10):

$$A_t = \int_{t_0}^t \lambda_s \, ds. \tag{3.2.1}$$

We obtained in Section 3 that integral $\int_{t_0}^t \lambda_s ds$ is an increasing function. The k'th firm's compensator writes similar to (3.2.1):

$$A_t^k = \int_{t_0}^t \lambda_s^k \, ds. \tag{3.2.2}$$

Because compensator A_t^k is a finite variation process, the difference $N_t^k - A_t^k$ is a martingale which implies:

$$\begin{aligned}
\mathbb{E}\left[N_{t+u}^k - A_{t+u}^k | \mathcal{F}_t\right] &= \mathbb{E}\left[N_{t+u}^k | \mathcal{F}_t\right] - \mathbb{E}\left[A_{t+u}^k | \mathcal{F}_t\right] \\
&= \mathbb{P}\left[t < \tau^k \leq t+u\right] - \mathbb{E}\left[\int_t^{t+u} \lambda_s^k \, ds | \mathcal{F}_t\right] \\
&\stackrel{*}{=} N_t^k - A_t^k = 0 \\
&\Rightarrow \mathbb{P}\left[t < \tau^k \leq t+u\right] = \mathbb{E}\left[\int_t^{t+u} \lambda_s^k \, ds | \mathcal{F}_t\right].
\end{aligned} \tag{3.2.3}$$

(*) *Martingale property* $\mathbb{E}\left[X_{t+u}|\mathcal{F}_t\right] = X_t, \forall t, u > 0.$

According to (3.0.12) and (3.0.13) we can write:

$$A_t = \sum_{k=1}^{m} A_t^k \Leftrightarrow \lambda_s = \sum_{k=1}^{m} \lambda_t^k. \tag{3.2.4}$$

By utilizing the Top down -approach, we can write according to the expression (3.1.5):

$$\lambda_t^k = Z_t^k \lambda_t. \tag{3.2.5}$$

With the expression (3.2.5) we can rewrite the expression (3.2.3) of the probability of default of the k'th firm in interval $(t, T]$:

$$\mathbb{P}\left[t < \tau^k \leq T\right] = \int_t^T \mathbb{E}\left[Z_s^k \lambda_s | \mathcal{F}_t\right] ds. \tag{3.2.6}$$

In expression (3.2.6) the idea of top-down approach is written concisely, i.e., with the choice of index intensity process and random thinning process we are able to calculate the default probabilities of the individual firms. However, the expected value inside integral (3.2.6) is challenging to compute in this form. Therefore, we need to rewrite this expected value.

If we rewrite thinning process (3.1.10) by denoting $M_t^{kn} = \mathbb{P}\left[\tau^k = T^n | \mathcal{F}_{t-}\right]$ we get:

$$Z^k(t) = \sum_{n=1}^{m} M_t^{kn} \mathbf{1}_{\{T^{n-1} < t \leq T^n\}}. \tag{3.2.7}$$

By substituting (3.2.7) to (3.2.6) we get:

$$\begin{aligned}\mathbb{P}\left[t < \tau^k \leq T\right] &= \sum_{n=1}^{m} \int_t^T \mathbb{E}\left[\lambda_s M_s^{kn} \mathbf{1}_{\{T^{n-1} < s \leq T^n\}} | \mathcal{F}_t\right] ds \\ &\overset{*}{=} \sum_{n=N_t+1}^{m} \int_t^T \mathbb{E}\left[\lambda_s M_s^{kn} \mathbf{1}_{\{N_s = n-1\}} | \mathcal{F}_t\right] ds. \end{aligned} \tag{3.2.8}$$

(*) *Event* $\left\{T^{n-1} < s \leq T^n\right\}$ *is equal to* $\{N_s = n-1\}$. *When this transformation is done, the sum start index need to change from one to* $N_t + 1$.

Expression (3.2.8) can be directly rewritten in the form:

$$\mathbb{P}\left[t < \tau^k \leq T\right] = - \sum_{n=N_t+1}^{m} \int_t^T \frac{\partial}{\partial z} \varphi_t \left(n - 1 - N_t, z, s, M_s^{kn} \lambda_s\right)|_{z=0} ds, \quad (3.2.9)$$

where the kernel writes:

$$\varphi_t(n, z, s, Y) = \mathbb{E}\left[e^{-zY} \mathbb{1}_{\{N_s - N_t = n\}} | \mathcal{F}_t\right]. \quad (3.2.10)$$

Thus, the remaining thing to do is to solve the expected value (3.2.10). Notice that the expressions (3.2.3), (3.2.6) and (3.2.9) describe the same probability, i.e., probability of default of the k'th firm to occur in a given time interval. When we directly wrote the probability with thinned intensity process in (3.2.6), the expected value inside integral is difficult to solve. We rewrote this expectations in terms of a kernel derivative in (3.2.9), which reduced the problem to solving kernel (3.2.10). It turns out, as we show later in this subsection, that the kernel φ can be efficiently solved via a time-frequency transformation, such as Fourier transform.

To simplify the calculation of kernel derivative $\partial \varphi / \partial z$ in (3.2.9), the smoothing of thinning process introduced in subsection 3.1 becomes useful. By introducing smoothed thinning process $\bar{Z}^k(t)$ as:

$$\bar{Z}^k(t) = \sum_{n=1}^{m} \bar{M}_t^{kn} \mathbf{1}_{\{T^{n-1} < t \leq T^n\}}, \quad (3.2.11)$$

where $\bar{M}_t^{kn} = \mathbb{P}\left[\tau^k = T^n | \bar{\mathcal{F}}_{t-}\right]$. The expression (3.2.9) becomes:

$$\mathbb{P}\left[t < \tau^k \leq T\right] = - \sum_{n=N_t+1}^{m} \int_t^T \frac{\partial}{\partial z} \varphi_t \left(n - 1 - N_t, z, s, \bar{M}_s^{kn} \bar{\lambda}_s\right)|_{z=0} ds, \quad (3.2.12)$$

By the property of the smoothed thinning process not observing the identity of defaulter at the default time, $\bar{M}^{kn}$ is independent of N^k. To evaluate the kernel derivative in (3.2.12) we can choose deterministic $\bar{M}^{kn}$ which simplifies the kernel derivative:

$$
\begin{aligned}
\frac{\partial}{\partial z}\varphi_t(n,z,s,\bar{M}_s^{kn}\bar{\lambda}_s) &= \frac{\partial}{\partial z}\mathbb{E}\left[e^{-z\bar{M}_s^{kn}\bar{\lambda}_s}\mathbb{1}_{\{N_s-N_t=n\}}|\bar{\mathcal{F}}_t\right] \\
&= \mathbb{E}\left[\frac{\partial}{\partial z}e^{-z\bar{M}_s^{kn}\bar{\lambda}_s}\mathbb{1}_{\{N_s-N_t=n\}}|\bar{\mathcal{F}}_t\right] \\
&= \mathbb{E}\left[-\bar{M}_s^{kn}\bar{\lambda}_s e^{-z\bar{M}_s^{kn}\bar{\lambda}_s}\mathbb{1}_{\{N_s-N_t=n\}}|\bar{\mathcal{F}}_t\right] \\
&\overset{*}{=} \bar{M}_s^{kn}\mathbb{E}\left[-\bar{\lambda}_s e^{-z\bar{M}_s^{kn}\bar{\lambda}_s}\mathbb{1}_{\{N_s-N_t=n\}}|\bar{\mathcal{F}}_t\right] \\
&= \bar{M}_s^{kn}\frac{\partial}{\partial z}\varphi_t(n,z,s,\bar{\lambda}_s). \qquad (3.2.13)
\end{aligned}
$$

(*) *We used the fact that $\bar{M}_s^{kn}$ was chosen to be deterministic, i.e. $\mathbb{E}\left[\bar{M}_s^{kn}|\bar{\mathcal{F}}_t\right] = \bar{M}_s^{kn}$.*

By (3.2.13) it suffices to evaluate $\frac{\partial}{\partial z}\varphi_t(n,z,s,\bar{\lambda}_s)$ to determine the probability $\mathbb{P}\left[t < \tau^k \leq T\right]$.

Next, we need to develop a computationally efficient way of determining the kernel derivative value at $z = 0$, i.e. $\frac{\partial}{\partial z}\varphi_t(n,z,s,\bar{\lambda}_s)|_{z=0}$. To do this, let us introduce counting process H with intensity ν such that $H_t \xrightarrow{t\to\infty} \infty$ (non-explosive property) and $\int_{t_0}^s \nu_t dt < \infty, \forall s > 0$ (integrability property). We define default counting process $N_t = \min\{H_t, m\} = H_t \wedge m$. The intensity λ of N satisfies $\lambda_t = \nu_t \mathbb{1}_{\{N_t<m\}}$. Now we can write expression (3.2.10) as:

$$
\varphi_t(n',z,s,Y) = \mathbb{E}\left[e^{-zY}\mathbb{1}_{\{H_s-H_t=n'\}}|\mathcal{F}_t\right], \quad n' < m - H_t, \qquad (3.2.14)
$$

or, since H_t is known given information $\mathcal{F}_t$, equivalently:

$$
\varphi_t(n,z,s,Y) = \mathbb{E}\left[e^{-zY}\mathbb{1}_{\{H_s=n\}}|\mathcal{F}_t\right], \quad n < m. \qquad (3.2.15)
$$

To easily compute expression (3.2.15), consider the following function:

$$
G(x,z,s,Y) = \mathbb{E}\left[e^{-zY}\mathbb{1}_{\{H_s\leq x\}}|\mathcal{F}_t\right], \quad x < m. \qquad (3.2.16)
$$

The derivative of $G(x,z,s,Y)$ with respect to counting variable x writes:

$$
\frac{\partial G(x,z,s,Y)}{\partial x} = \mathbb{E}\left[e^{-zY}\delta_{\{H_s=x\}}|\mathcal{F}_t\right], \quad x < m, \qquad (3.2.17)
$$

where $\delta_{\{H_s=x\}} = \begin{cases}\infty, & H_s = x\\ 0, & H_s \neq x\end{cases}$ is the dirac delta.

Next, let us introduce Fourier transform pair. The strategy is to transform function G in (3.2.16) from count domain $x \in [H_t, m]$ to frequency domain $u \in \mathbb{R}$, solve the expected value in frequency domain, and transform the expression back in the count domain. The Fourier transform of function $f(x)$ to $F(u)$ and inverse Fourier transform of function $F(u)$ to $f(x)$, denoted by $\mathscr{F}\{f(x)\}(x,u)$ and $\mathscr{F}^{-1}\{F(u)\}(u,x)$, respectively, are given as:

$$F(u) = \mathscr{F}\{f(x)\}(x,u) = \int_{\mathbb{R}} e^{i2\pi ux} f(x)dx \tag{3.2.18}$$

$$f(x) = \mathscr{F}^{-1}\{F(u)\}(u,x) = \int_{\mathbb{R}} e^{-i2\pi ux} F(u)du \tag{3.2.19}$$

Note that consecutive Fourier transform and inverse Fourier transform preserve the original function:

$$\begin{aligned}
\mathscr{F}^{-1}\{F(u)\}(u,x) &= \int_{\mathbb{R}} e^{-i2\pi ux} F(u)du \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i2\pi ux} e^{i2\pi ux'} f(x')dx'du \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i2\pi u(x-x')} f(x')dx'du \\
&\overset{*}{=} \int_{\mathbb{R}} \delta(x-x') f(x')dx' \\
&= f(x).
\end{aligned}$$

(*) *We use the fact* $\hat{\delta}(u) = \mathscr{F}\{\delta(x-x')\}(x,u) = \int_{\mathbb{R}} e^{i2\pi ux}\delta(x-x')dx = e^{i2\pi ux'}$. *The inverse transform becomes* $\delta(x-x') = \mathscr{F}^{-1}\left\{\hat{\delta}(u)\right\}(u,x) = \int_{\mathbb{R}} e^{-i2\pi ux} e^{i2\pi ux'}du = \int_{\mathbb{R}} e^{-i2\pi u(x-x')}du$.

Note that function G is an increasing function with respect to x, thus we can treat G as a measure. This implies that we use the Fourier-Stieltjes transform which reduces to ordinary Fourier transform pair (3.2.18)-(3.2.19) due to identity (3.2.17):

$$
\begin{aligned}
\mathcal{G}(u,z,s,\nu_s) = \mathscr{F}\{G(x,z,s,\nu_s)\}(x,u) &= \int_{\mathbb{R}} e^{i2\pi ux} dG(x,z,s) \\
&= \int_{\mathbb{R}} e^{i2\pi ux} \mathbb{E}\left[e^{-z\nu_s}\delta\left(x-H_s\right)\right] dx \\
&= \mathbb{E}\left[e^{-z\nu_s}\int_{\mathbb{R}} e^{i2\pi ux}\delta\left(x-H_s\right) dx\right] \\
&= \mathbb{E}\left[e^{-z\nu_s}e^{i2\pi uH_s}\right] \\
&= \mathbb{E}\left[e^{i2\pi uH_s - z\nu_s}\right].
\end{aligned} \tag{3.2.20}
$$

In order to solve the expression (3.2.20), we need to choose the dynamics for the diffusion process $\nu(t)$. In subsection 3.3 we consider the choice of the dynamics in detail. Let us for now, choose the following diffusion process for the intensity parameter ν:

$$
d\nu_t = \kappa\left(c-\nu_t\right)dt + \delta dJ_t, \tag{3.2.21}
$$

where $\kappa \geq 0$, $c > 0$, $\delta \geq 0$, $\nu_{t_0} > 0$ and $J = lH$ with constant $l \in (0,1]$. The interpretation of parameters is that c is the equilibrium intensity towards which the intensity ν_t moves. The intensity can be distorted from c by a credit event, which increases the intensity. After the credit event, the intensity moves towards c at rate κ.

With the choice of (3.2.21) for the diffusion process of the intensity parameter ν, it can be shown (see Appendix A, Proposition 1. in [12], additionally [6] and [13]) that expression (3.2.19) can be written under affine structure as:

$$
\mathbb{E}\left[e^{i2\pi uH_s - z\nu_s}\right] = e^{A(s)+B(s)\nu_{t_0}}, \tag{3.2.22}
$$

where coefficient functions A and B satisfy the following ODEs:

$$
\frac{\partial}{\partial s}A(s) = \kappa c B(s), \tag{3.2.23}
$$

$$
\frac{\partial}{\partial s}B(s) = -\kappa B(s) - 1 + e^{i2\pi u + \delta l B(s)}, \tag{3.2.24}
$$

with boundary conditions $B(t_0) = -z$ and $A(t_0) = 0$. Function G can be recovered by inverse Fourier transform:

$$
\begin{aligned}
\frac{dG(x', z, s)}{dx'} &= \mathscr{F}^{-1}\{\mathcal{G}(u, z, s, \nu_s)\}(u, x') \\
&= \int_{-\infty}^{\infty} e^{-i2\pi u x'} \mathbb{E}\left[e^{i2\pi u H_s - z\nu_s}\right] du \\
&= \mathbb{E}\left[e^{-z\nu_s} \int_{-\infty}^{\infty} e^{-i2\pi u\left(x' - H_s\right)} du\right] \\
&= \mathbb{E}\left[e^{-z\nu_s} \delta\left(x' - H_s\right)\right].
\end{aligned} \tag{3.2.25}
$$

From (3.2.25) we can recover $G(x, z, s, \nu_s)$:

$$
\begin{aligned}
& dG(x', z, s, \nu_s) = \mathbb{E}\left[e^{-z\nu_s} \delta\left(x' - H_s\right)\right] dx' \\
& \Rightarrow \int_{-\infty}^{x} dG(x', z, s, \nu_s) = \mathbb{E}\left[\int_{-\infty}^{x} e^{-z\nu_s} \delta\left(x' - H_s\right) dx'\right] \\
& \Leftrightarrow G(x, z, s, \nu_s) - \underbrace{\left[\lim_{\mu \to -\infty} G(\mu, z, s, \nu_s)\right]}_{=0} = \mathbb{E}\left[e^{-z\nu_s} \mathbb{1}_{\{H_s \leq x\}}\right].
\end{aligned} \tag{3.2.26}
$$

From expression (3.2.26) by setting $z = 0$, we directly get the probability that default counting process $N = H \wedge m$ is less than x:

$$
\mathbb{P}\left[N_s \leq x\right] = \mathbb{E}\left[\mathbb{1}_{\{N_s \leq x\}}\right] = G(x, 0, s, \nu_s), \quad x < m. \tag{3.2.27}
$$

Next, we aim to do the random thinning given the index intensity process ν. The modeling primitive in the thinning is the choice of constituent default probabilities $\bar{M}_t^{kn} = \mathbb{P}\left[\tau^k = T^n | \bar{\mathcal{F}}_{t-}\right]$ according to expression (3.2.11). After this choice, it remains to solve the kernel derivative $\frac{\partial}{\partial z}\varphi_t(n, z, s, \bar{\lambda}_s)$.

To solve the kernel derivative in (3.2.9), observe that:

$$
\frac{\partial}{\partial z} G(x, z, s) = \frac{\partial}{\partial z} \mathbb{E}\left[e^{-z\nu_s} \mathbb{1}_{\{H_s \leq x\}}\right] = \mathbb{E}\left[-\nu_s e^{-z\nu_s} \mathbb{1}_{\{H_s \leq x\}}\right]. \tag{3.2.28}
$$

Thus $\mathcal{G}_z(u, z, s) = \mathscr{F}\left\{\frac{\partial}{\partial z} G(x, z, s)\right\}(x, u) = \frac{\partial}{\partial z}\mathcal{G}(u, z, s) = \mathbb{E}\left[-\nu_s e^{i2\pi u H_s - z\nu_s}\right]$. By (3.2.22) we get:

$$
\begin{aligned}
\mathcal{G}_z(u, z, s) &= \mathbb{E}\left[-\nu_s e^{i2\pi u H_s - z\nu_s}\right] \\
&= \left(A_z(s) + B_z(s)\nu_{t_0}\right) e^{A(s) + B(s)\nu_{t_0}} \\
&= \left(A_z(s) + B_z(s)\nu_{t_0}\right) \mathcal{G}(u, z, s).
\end{aligned} \tag{3.2.29}
$$

Setting $z = 0$, in (3.2.29) yields $\mathcal{G}_z(u, z, s)|_{z=0} = (A_z(s)|_{z=0} + B_z(s)\nu_{t_0}|_{z=0})\,\mathcal{G}(u, 0, s)$, where coefficient functions $A_z(s)$ and $B_z(s)$ can be solved by using system (3.2.23)-(3.2.24):

$$\frac{\partial}{\partial s}A_z(s) = \kappa c B_z(s), \tag{3.2.30}$$
$$\frac{\partial}{\partial s}B_z(s) = -\kappa B_z(s) + e^{i2\pi u + \delta l B(s,z=0)}\delta l B_z(s), \tag{3.2.31}$$

with initial conditions $B_z(t_0) = -1$ and $A_z(t_0) = 0$. Function G_z is recovered by the inverse Fourier transform:

$$\begin{aligned}
\frac{dG_z(x', 0, s)}{dx'} &= \mathscr{F}^{-1}\{\mathcal{G}_z(u, z, s, \nu_s)|_{z=0}\}(u, x') \\
&= \int_{-\infty}^{\infty} e^{-i2\pi u x'}\mathbb{E}\left[-\nu_s e^{i2\pi u H_s}\right] du \\
&= \mathbb{E}\left[-\nu_s \int_{-\infty}^{\infty} e^{-i2\pi u\left(x' - H_s\right)} du\right] \\
&= \mathbb{E}\left[-\nu_s \delta\left(x' - H_s\right)\right].
\end{aligned} \tag{3.2.32}$$

From (3.2.32) we can recover $G_z(x, 0, s, \nu_s)$:

$$\begin{aligned}
&dG_z(x', 0, s, \nu_s) = \mathbb{E}\left[-\nu_s \delta\left(x' - H_s\right)\right] dx' \\
&\Rightarrow \int_{-\infty}^{x} dG_z(x', 0, s, \nu_s) = \mathbb{E}\left[-\nu_s \int_{-\infty}^{x} \delta\left(x' - H_s\right) dx'\right] \\
&\Leftrightarrow G_z(x, 0, s, \nu_s) - \underbrace{\left[\lim_{\mu \to -\infty} G_z(\mu, 0, s, \nu_s)\right]}_{=0} = \mathbb{E}\left[-\nu_s \mathbb{1}_{\{H_s \le x\}}\right].
\end{aligned} \tag{3.2.33}$$

To use the result (3.2.33) to solve the kernel derivative $\frac{\partial}{\partial z}\varphi_t(n, z, s, \nu_s)|_{z=0} = \mathbb{E}\left[-\nu_s \mathbb{1}_{\{H_s = n\}}|\mathcal{F}_t\right]$, $n < m$, we need to identify the jump point of $G_z(x, 0, s, \nu_s)$ from 0. The jump point $x = n$ gives the kernel derivative value at time s. To solve the constituent default probabilities $\mathbb{P}\left[t < \tau^k \le T\right]$, it only remains to integrate the kernel derivative over time interval $[t, T]$.

Now, we are set to determine the constituent default probabilities $\mathbb{P}\left[t < \tau^k \le T\right]$ and default counting process probabilities $\mathbb{P}\left[N_s \le x\right]$, as written in expressions (3.2.12) and (3.2.27), respectively. It suffices to consider only these probabilities when pricing credit derivatives.

Before we can use the model developed in this subsection for pricing, we need to calibrate the model. The calibration is a procedure to find the free parameters of the model optimally, where optimal is defined in terms of an objective function. Traditionally, the objective function is a norm between model implied derivative prices and market observed derivative prices. In this context, the calibration is a minimization problem where we minimize the norm with respect to the free parameters. The free parameters are determined by the choice of the intensity process ν_t and random thinning elements $\bar{M}_s^{kn}$. For example, in the intensity process definition (3.2.21), the free parameters are κ, c and δ. We consider the calibration in detail in Section 4.

As we have now developed the method of constructing index and index constituent default probabilities, let us use the developed methods in example environments. This is done in the subsections 3.2.1 and 3.2.2.

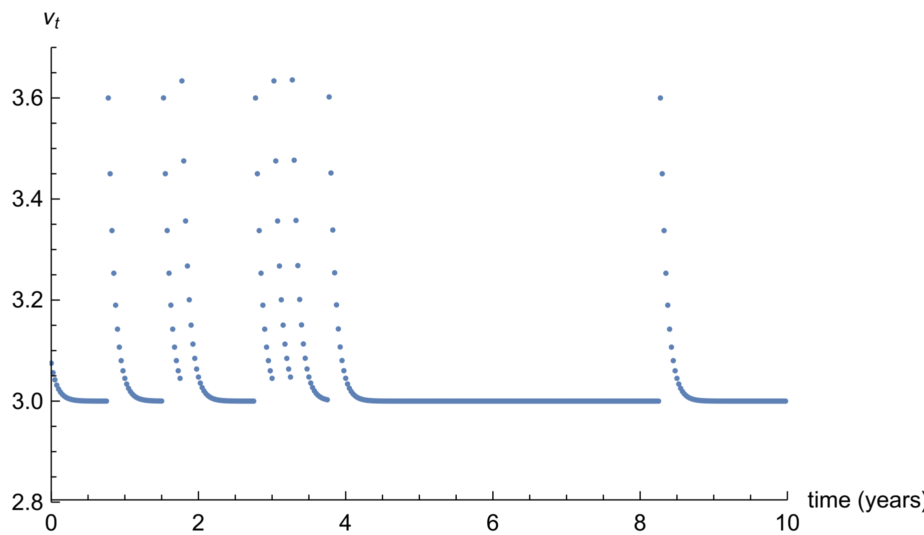

Figure 3: Example of realized diffusion process ν_t with parameter values $\kappa = 1.0$, $c = 3.0$, $\delta = 1.0$, $\nu_{t_0} = 3.1$, and $l = 0.6$

3.2.1 Example simulation of intensity based default counting process with basket of 20 firms

Let us model the default probabilities $\mathbb{P}[N_s \leq x]$ under intensity process (3.2.21) with system parameters $\kappa = 1.0$, $c = 3.0$, $\delta = 1.0$, $\nu_{t_0} = 3.1$, and $l = 0.6$. We have $m = 20$ firms in the system and we analyze system behavior in time interval $[0, T]$, where $T = 10$ years. With these parameter choices, the realized diffusion process ν_t looks similar to Figure 3. Note from Figure 3 that the diffusion process starts at rate 3.1, and stabilizes towards 3.0 at rate 1. When credit event occurs, the intensity jumps at rate 0.6. After jump the process aims toward 3.0.

In Figure 4, the default counting process probabilities $\mathbb{P}[N_t \leq x]$ for given $x \in \{1, ..., 20\}$ are shown in time interval $t \in [0, T]$. The probabilities are solved via the Fourier-transform pair (3.2.20) and (3.2.25). In Figure 6 the counting process N_t is thinned to constituent probabilities $\mathbb{P}\left[0 < \tau^k \leq T\right]$. The thinning is done by using thinning matrix $\bar{\mathbf{M}}$ with elements $\bar{M}_{\underline{t}}^{kn} = \mathbb{P}\left[\tau^k = T^n | \bar{\mathcal{F}}_{t-}\right]$ shown in Figure 5. Note that row and column sums of $\bar{\mathbf{M}}$ sum up to one.

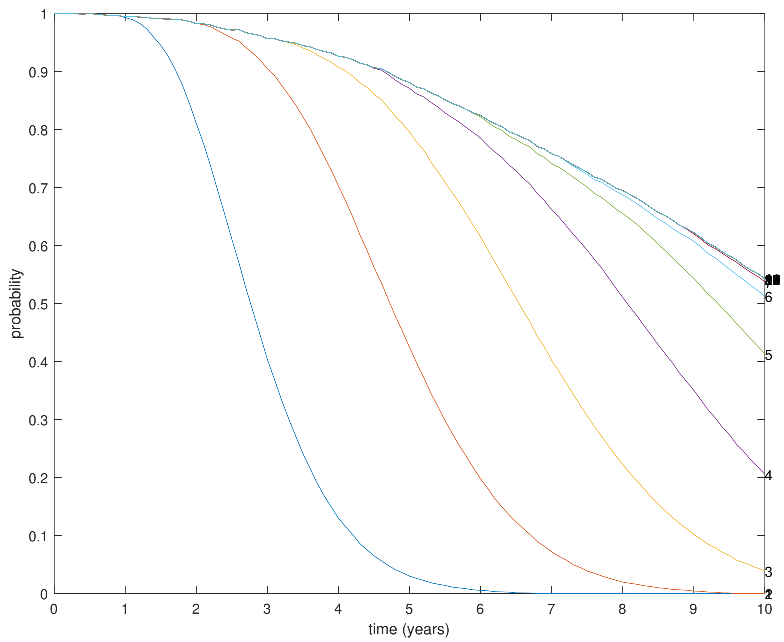

Figure 4: Default counting process probabilities $\mathbb{P}[N_t \leq x]$ with parameter values $\kappa = 1.0$, $c = 3.0$, $\delta = 1.0$, $\nu_{t_0} = 3.1$, and $l = 0.6$. The number at right side is the value of x.

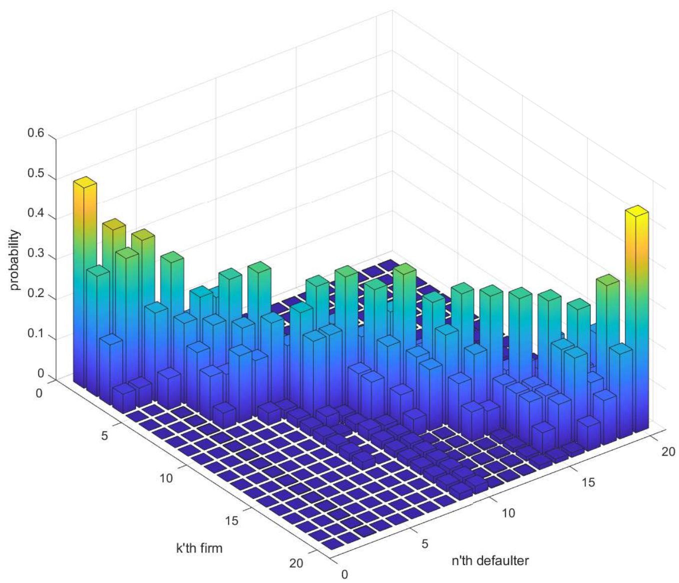

Figure 5: Thinning matrix M with elements $\bar{M}_t^{kn} = \mathbb{P}\left[\tau^k = T^n | \bar{\mathcal{F}}_{t-}\right]$.

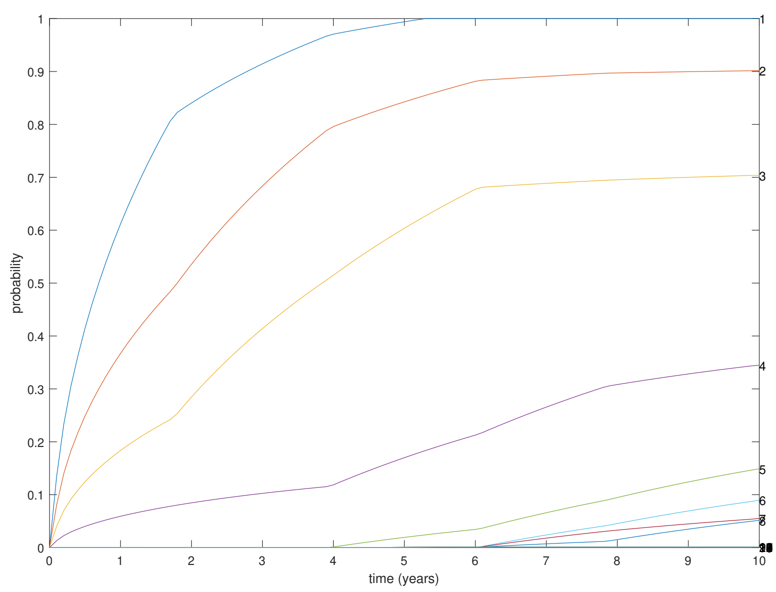

Figure 6: Default probabilities $\mathbb{P}\left[0 < \tau^k \leq T\right]$ with parameter values $\kappa = 1.0$, $c = 3.0$, $\delta = 1.0$, $\nu_{t_0} = 3.1$, and $l = 0.6$. The number at right side is the identifier of a specific firm (i.e. k). The thinning matrix elements $\bar{M}_t^{kn}$ are set up so that as k increases the riskiness of the firm approximately increases.

3.2.2 Example simulation of intensity based default counting process with basket of 3 firms

Let us next consider a basket of three firms. In this example we work on the numeric values, instead of figures. The parameter initialization of the system is done below:

(i) Number of firms $m = 3$ with equal weights $w^k = 1/3, \forall k \in \{1, 2, 3\}$.

(ii) Recovery rates $R^k = 0.3, \forall k \in \{1, 2, 3\}$. Thus, loss rates are $l^k = 0.7, \forall k \in \{1, 2, 3\}$.

(iii) The start date of the contract is the 12th of July 2019. The length of the contract is approximately 2 years, ending the 20th of June 2021.

(iv) Premium leg pays quarterly coupon on IMM dates.

(v) Thinning matrix M_{t_0} is given as:

$$M = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \tag{3.2.34}$$

(vi) Intensity process is assumed to be as in expression (3.2.21) with parameter values $\kappa = 1$, $c = 0.5$, $\delta = 0.2$ and $\nu_{t_0} = 0.5$.

(vii) Discounting is done using USD ISDA standard curve, which is on the 12th of July 2019 as follows:

USD ISDA standard curve (%)					
1 month	2 month	3 month	6 month	1 year	2 year
2.3250	2.3248	2.3034	2.2126	2.1933	1.9060

With the contract specification (i)-(vii), the fair spread is calculated to be $S_F(t_0, T) = 7.47\%$. This is done by equating the expected discounted value of the premium and protection legs. At the start date t_0 the default counting process probabilities $\mathbb{P}[N_t < x]$ with respect to the time from start date are as shown in the table below:

Default counter probabilities $\mathbb{P}[N_t < x]$ as a function of time from start t_0.						
	0 months	3 months	6 months	9 months	1 year	maturity
$\mathbb{P}[N_t < 1]$	1.0000	0.9221	0.8625	0.8165	0.7805	0.6983
$\mathbb{P}[N_t < 2]$	1.0000	0.9533	0.9175	0.8899	0.8683	0.8190
$\mathbb{P}[N_t < 3]$	1.0000	0.9844	0.9725	0.9633	0.9561	0.9397

According to the expression (3.3.4), the probabilities can be written as $\mathbb{P}[N_t = x]$:

Default counter probabilities $\mathbb{P}[N_t = x]$ as a function of time from start t_0.					
	0 months	3 months	6 months	9 months	1 year	maturity
$\mathbb{P}[N_t = 0]$	1.0000	0.9221	0.8625	0.8165	0.7805	0.6983
$\mathbb{P}[N_t = 1]$	0.0000	0.0312	0.0550	0.0734	0.0878	0.1207
$\mathbb{P}[N_t = 2]$	0.0000	0.0311	0.0550	0.0734	0.0878	0.1207
$\mathbb{P}[N_t = 3]$	0.0000	0.0156	0.0275	0.0367	0.0439	0.0603

By perfoming random thinning with the random thinning matrix M given in (v), we get the following constituent default probabilities $\mathbb{P}\left[0 < \tau^k \le t\right]$:

Probabilities $\mathbb{P}\left[0 < \tau^k \le t\right]$ as a function of time from start t_0.						
	0 months	3 months	6 months	9 months	1 year	maturity
$\mathbb{P}\left[t_0 < \tau^1 \le t\right]$	0.0000	0.0835	0.1605	0.2329	0.3019	0.5576
$\mathbb{P}\left[t_0 < \tau^2 \le t\right]$	0.0000	0.0239	0.0459	0.0666	0.0863	0.1593
$\mathbb{P}\left[t_0 < \tau^3 \le t\right]$	0.0000	0.0119	0.0229	0.0333	0.0431	0.0797

Let us assume that after 9 months from the basket inception, firm $k = 1$ defaults with recovery rate 30%. This means that our basket has suffered a loss of $w^1 * l^1 = 1/3 * 0.7 = 7/30 \approx 0.2333$, and for the remaining lifetime it only consists of two firms $k = 2$ and $k = 3$. The intensity process will jump up from the equilibrium value $c = 0.5$ by $\delta * l = 0.2 * 0.7 = 0.14$. Therefore, starting from the 9 months after the basket inception, we have the intensity process parameters $\kappa = 1$, $c = 0.5$, $\delta = 0.2$ and $\nu_{t_{9m}} = 0.64$. Starting from the 9 months after basket inception, the default counting probabilities are:

Default counter probabilities $\mathbb{P}[N_t < x]$ as a function of time from t_{9m}.						
	9 months	1 year	1 year, 3 months	1 year, 6 months	1 year, 9 months	maturity
$\mathbb{P}[N_t < 1]$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\mathbb{P}[N_t < 2]$	1.0000	0.9207	0.8658	0.8266	0.7978	0.7762
$\mathbb{P}[N_t < 3]$	1.0000	0.9736	0.9553	0.9422	0.9326	0.9254

Let us assume that after the default of the firm $k = 1$, the thinning matrix $M_{t_{9m}}$ is given as:

$$M = \begin{bmatrix} 0.55 & 0.45 \\ 0.45 & 0.55 \end{bmatrix}. \tag{3.2.35}$$

The constituent default probabilities $\mathbb{P}\left[0 < \tau^k \le t\right]$ become:

Probabilities $\mathbb{P}\left[0 < \tau^k \leq t\right]$ as a function of time from t_{9m}.						
	9 months	1 year	1 year, 3 months	1 year, 6 months	1 year, 9 months	maturity
$\mathbb{P}\left[t_0 < \tau^1 \leq t\right]$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\mathbb{P}\left[t_0 < \tau^2 \leq t\right]$	0.0000	0.0813	0.1527	0.2176	0.2780	0.3352
$\mathbb{P}\left[t_0 < \tau^3 \leq t\right]$	0.0000	0.0665	0.1250	0.1780	0.2274	0.2743

After the default of firm $K = 1$ at 9 months from inception with this specification we have fair spread $S_F(t_{9m}, T) = 9.7\%$.

3.3 Choosing the intensity process

As we noted in the subsection 3.2, the pricing model properties are determined by the compensator of the default counting process, and the compensator is uniquely defined by the intensity process. Therefore, the choice of intensity process ν_t dictates how the model prices credit derivatives.

In this subsection, we consider various choices of intensity process. It is important to notice that all the diffusion dynamics that we consider belong into so-called affine diffusion processes. The definition and examples of affine processes are exhaustively discussed in [8]. The concise definition is that the drift of ν_t, local variance of ν_t and jump intensity of ν_t have affine dependence of ν_t, i.e., they can be written as a linear combination of ν_t. If the intensity process fulfills this requirement, we can solve expected values via system of ordinary differential equations, as in (3.2.22)-(3.2.24).

In the subsection 3.2 we assumed the compensator intensity process to follow Hawkes jump diffusion process (see [14]) as in expression (3.2.21). Note that in this case the drift can be written as $\kappa c - \kappa\nu_t$, local variance as 0, which are both linear combinations of ν_t. If we in addition assume that jump intensity of J_t is a linear combination of ν_t, i.e., of form $a + b\nu_t$, $a, b \in \mathbb{R}$, the intensity process ν_t is affine process. Therefore, we are able to derive system of two differential equations (3.2.23)-(3.2.24) that solve the kernel derivative (3.2.10) that was used to calculate the probability of defaults of CDS contract constituents.

Based on the derivations in subsection 3.2, it is straightforward to consider other intensity processes. In the most simple setting, we can set intensity ν_t to be constant, i.e. $\nu_t = c$. This type of intensity process is known as Poisson process, and by requiring that:

$$\nu_t^k = \zeta^k e^{-\zeta^k t}, \tag{3.3.1}$$

where $\zeta \in \mathbb{R}$, we have according to (3.2.3), that the default times of constituent firms follow exponential distribution with parameter ζ^k:

$$\begin{aligned}
\mathbb{P}\left[t < \tau^k \leq t+u\right] &= \mathbb{E}\left[\int_t^{t+u} \nu_s^k ds | \mathcal{F}_t\right] \\
&= \int_t^{t+u} \zeta^k e^{-\zeta^k s} ds \\
&= e^{-\zeta^k t} - e^{-\zeta^k (t+u)}.
\end{aligned} \tag{3.3.2}$$

Based on expression (3.0.13), we have that $c = \sum_{k=1}^{m} \zeta^k e^{-\zeta^k t}$. Despite the simplicity of the Poisson process, it is rarely used in the modeling of the default counting processes. This is because of the fact that it does not address the correlation between constituent firms, and the state of the economy in general. To alleviate such problems, we could make set $\{\zeta_t^k\}_{k=1}^m$ to vary based on some economic conditions, but the problem with this approach is how to determine the state of the economy in the future. The state of the economy in the future is a set of stochastic variables, which requires us to interpret c as a stochastic variable.

One simple way to extend the Poisson process is to model intensity ν as:

$$\nu_t = c + \sum_{k=1}^{K} d^k Z_t^k, \tag{3.3.3}$$

where $\{Z_t^k\}_{k=1}^K$ are random variables modeling the state of the economy, and $\{d_t^k\}_{k=1}^K$ are weights, that account for the strength of the impact that variables Z have on the intensity. One such a process is compound birth loss process, which writes:

$$\nu_t = c + \delta J_t, \tag{3.3.4}$$

where $\delta \geq 0$, $J_t = lH$, H is a jump process with jumps bounded to one, and l is the loss rate. The expression (3.3.4) increases intensity by $\delta * l$ every time there occurs a credit event in the basket of constituent firms. This increases the probability of default of constituent firms every time there is a credit event in the basket of firms, and therefore is a simple way to addresses the correlation between constituent firms.

To further extend our intensity model, we can assume that there is some equilibrium value for ν_t towards which the intensity tends to. If intensity is disturbed from the equilibrium value, it tends back towards the equilibrium value at a certain rate. At the most simple setting, such an intensity process writes in differential form:

$$d\nu_t = \kappa(c - \nu_t)dt + \delta dJ_t, \tag{3.3.5}$$

where c is possibly a stochastic variable. In the subsection 3.2 we used the process (3.3.5) with deterministic c. By denoting that the expression (3.3.5) is of form $y' + p(t)y = q(t)$, we can write the solution in the form $y = \frac{1}{\mu(t)}(\int_{t_0}^t \mu(s)q(s)ds + c)$, where $\mu(t) = e^{\int_{t_0}^t p(s)ds}$. This gives intensity process:

$$\nu_t = \nu_{t_0} e^{-\kappa(t-t_0)} + ce^{-\kappa t_0} + \delta \int_{t_0}^t e^{-\kappa(t-s)} dJ_s. \tag{3.3.6}$$

To generalize the intensity process even further, we can introduce a random component that disturbs the intensity ν_t between times of the credit events in the counting process H. One such model is a Cox-Ingersoll-Ross type process:

$$d\nu_t = \kappa(c - \nu_t)dt + \sigma\sqrt{\nu_t}dW_t + \delta dJ_t, \tag{3.3.7}$$

where W_t is a standard Brownian motion. Notice that the local variance of (3.3.7) writes $\sigma^2\nu_t$, thus ν_t remains affine. The process (3.3.7) can be solved in a similar fashion as the process (3.3.5), and it writes:

$$\nu_t = \nu_{t_0}e^{-\kappa(t-t_0)} + ce^{-\kappa t_0} + \sigma\int_{t_0}^{t} e^{-\kappa(t-s)}\sqrt{\nu_s}dW_s + \delta\int_{t_0}^{t} e^{-\kappa(t-s)}dJ_s. \tag{3.3.8}$$

In the Figure 10, the intensity processes (3.3.4), (3.3.5) and (3.3.7) are plotted with example realizations.

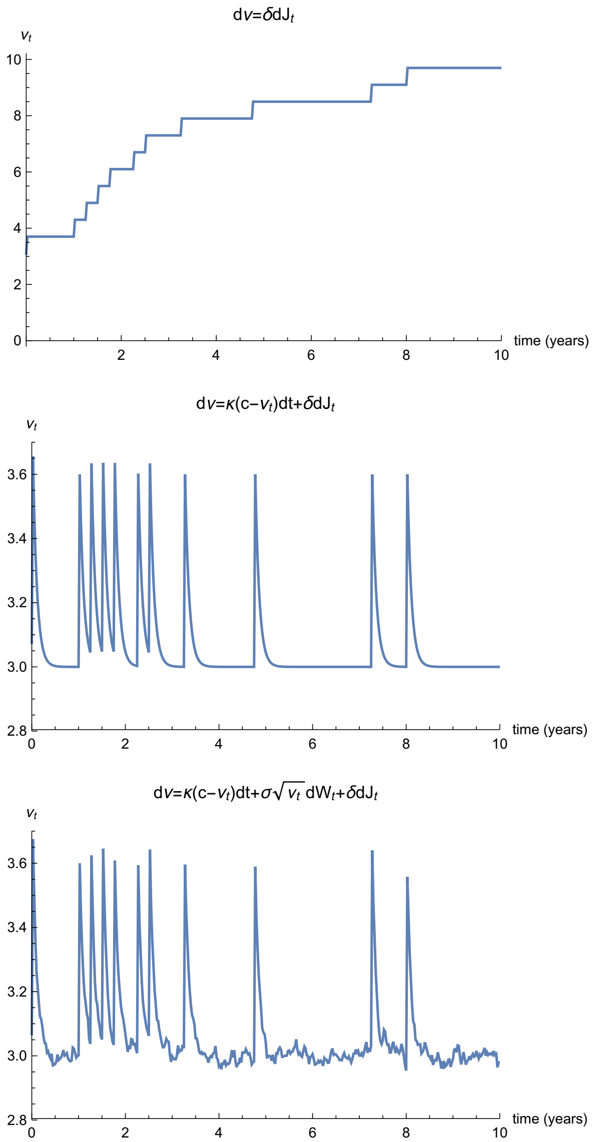

Figure 7: Example realizations of the intensity processes (3.3.4), (3.3.5) and (3.3.7).

4 Calibration of intensity based default counting processes

The pricing of credit derivatives with the models derived in Section 3 is a two-step process: first we need to calibrate the model, and then apply the calibrated model to the product on hand. The calibration is either a minimization problem or a root-solving exercise, depending on the chosen calibration scheme. The calibration is done using market quotes on chosen derivative instruments, and the aim is to match the model implied price of the calibration products exactly or closely to the market observed quotes.

Let us next consider the calibration of the intensity based model derived in subsection 3.2. This means optimally finding the parameters $\theta = \{\kappa, c, \delta, \nu_0\}$, where $\kappa \geq 0, c > 0, \delta \geq 0,\ \nu_0 > 0$. Optimal can be defined as equating the model implied spread/upfront payment to market spread/upfront payment at single time instance:

$$\text{Model}(t_j, \theta) = \text{Mid}(t_j), \tag{4.0.1}$$

or as finding a solution to minimization problem, where we minimize the error between model implied and market spread/upfront payments in the time interval $[t_0, t_J]$:

$$\min_{\theta \in \Theta} \left\{ \sum_{j=0}^{J} \frac{(\text{Mid}(t_j) - \text{Model}(t_j, \theta))^2}{\text{Mid}(t_j)} \right\}, \tag{4.0.2}$$

where $\text{Mid}(t_j)$ is the mid-price of the instrument at time t_j, and $\text{Model}(t_j, \theta)$ is the price of the instrument at time t_j implied by the pricing model that is based on the probability curves from subsection 3.2. The parameter space θ can be constrained to a feasible set Θ that is chosen based on the contract type. For the root finding problems of form (4.0.1) we can use methods such as bisection and Newton's method, and for the minimization problems of form (4.0.2) we can use gradient based methods such as gradient descent.

It is important to notice that the model implied spread/upfront payment, protection leg value and premium leg value are not monotonic functions of the parameter space. This is verified by simulations that are performed on the linear CDS index contract, i.e., tranche contract with attachment of 0% and detachment of 100%. The simulation is done on the Markit CDX North America HY s32 index with market quoted values given in Table 2. Simulation results are presented in Figures 7 and 8.

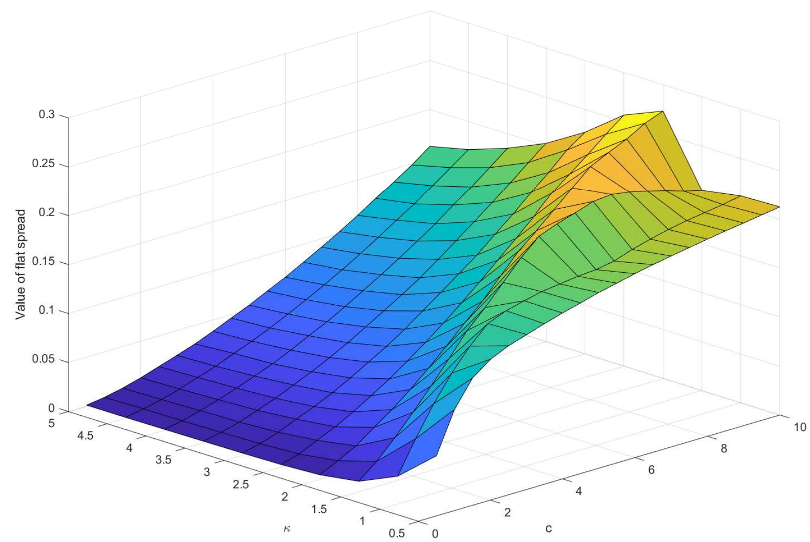

Figure 8: Dependence of flat spread $S_F(t_0, t_N)$ on parameters $\kappa \in [0.5, 5.0]$, $c \in [0.5, 10]$ and $\nu_{t_0} = c$ with constant parameter values $\delta = 2.0$ and $l = 0.7$

Table 1: Markit CDX HY s32 5y (20-Jun-2024) Ref 376.3346

Start date	End date	Coupon days	Mid	Quote
30-May-2019	20-Jun-2024	20-Jun-2019 to 20-Jun-2024 on IMM dates	3.763346%	Spread, 30% recovery

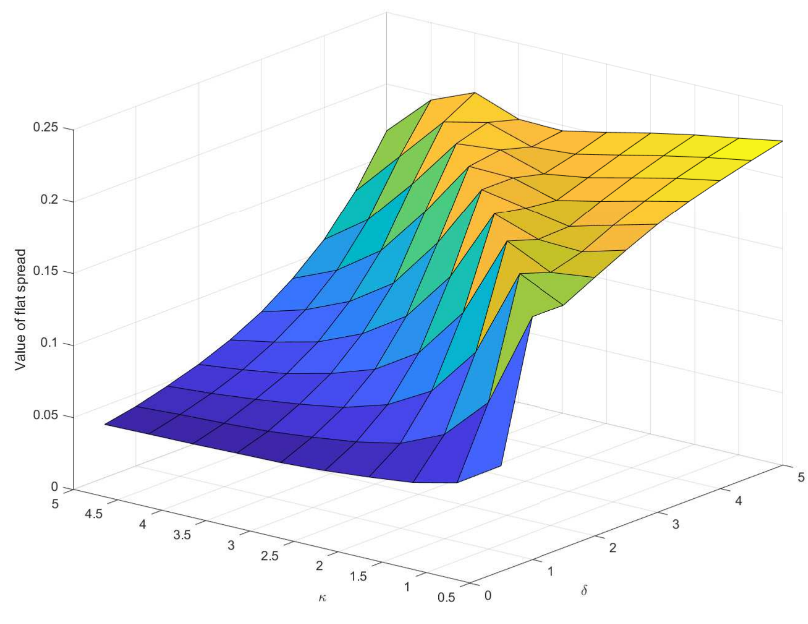

Figure 9: Dependence of flat spread $S_F(t_0, t_N)$ on parameters $\kappa \in [0.5, 5.0]$, $\delta \in [0.5, 5.0]$ and $\nu_{t_0} = c$ with constant parameter values $c = 5$ and $l = 0.7$

To address the gradient based optimization methods, let us consider the derivative of the default counting process with respect to the intensity process parameters θ. For all the CDS contracts discussed in Section 2, the survival probability/process is written in terms of the default counting process N, according to expression (3.0.2). For example, the expected value of CDS index loss process function $L^{(index)}(t)$ writes:

$$\mathbb{E}[L^{(index)}(t)] = \frac{1}{m}\sum_{k=1}^{m} k * \mathbb{P}\left[N_t = k \,| N_{t_0}\right]. \tag{4.0.3}$$

In the subsection 3.2, we have obtained the probabilities that default counting process N_t is less than given $x \in \{1, ..., m\}$, i.e., $\mathbb{P}\left[N_t \leq x | N_{t_0}\right]$. With this information we can write:

$$\begin{aligned} \mathbb{P}\left[N_t = k \,| N_{t_0}\right] &= \mathbb{P}\left[k \leq N_t < k+1 | N_{t_0}\right] \\ &= \begin{cases} 1 - \mathbb{P}\left[N_t \leq m | N_{t_0}\right], & k = m \\ \mathbb{P}\left[N_t \leq k+1 | N_{t_0}\right] - \mathbb{P}\left[N_t \leq k | N_{t_0}\right], & k < m. \end{cases} \end{aligned} \tag{4.0.4}$$

The sensitivity of the expected loss process value $\mathbb{E}[L^{(index)}(t)]$ with respect to variable $x \in \theta$:

$$\frac{\partial}{\partial x}\mathbb{E}[L^{(index)}(t)] = \frac{1}{m}\sum_{k=1}^{m} k * \frac{\partial}{\partial x}\mathbb{P}\left[N_t = k \,| N_{t_0}\right]. \tag{4.0.5}$$

Based on the expression (4.0.5), the sensitivity of the tranche loss process with respect to c, i.e, $\frac{\partial}{\partial x}\mathbb{E}[L^{(tranche)}(t)]|_{c=c_0}$, can be written as:

$$\frac{\partial}{\partial x}\mathbb{E}[L^{(tranche)}(t)]|_{c=c_0} = \frac{1}{mK_2}\sum_{k=1}^{m} k * \mathbb{1}_{\left\{K_1 < \frac{k}{m} < K_2\right\}} \frac{\partial}{\partial x}\mathbb{P}\left[N_t = k \,| N_{t_0}\right]. \tag{4.0.6}$$

The derivatives of the default counter probabilities $\mathbb{P}\left[N_t = k \,| N_{t_0}\right]$ are splitted to two cases, $k < m$ and $k = m$. Let us consider these two cases separately:

$k = m$:

$$
\begin{aligned}
&\frac{\partial}{\partial x}\mathbb{P}\left[N_t = m \,|N_{t_0}\right] = -\frac{\partial}{\partial x}\mathbb{P}\left[N_t \leq m|N_{t_0}\right] = -\frac{\partial}{\partial x}G\left(m, 0, s, \nu_s\right) \\
&= -\frac{\partial}{\partial x}\int_{-\infty}^{m} \mathscr{F}^{-1}\left\{\mathcal{G}\left(u, 0, s, \nu_s\right)\right\}(u, x')dx' \\
&= -\int_{-\infty}^{m}\int_{-\infty}^{\infty} e^{-i2\pi ux'}\left(\frac{\partial}{\partial x}e^{A(s)+B(s)\nu_{t_0}}\right)dudx' \\
&= -\int_{-\infty}^{m}\int_{-\infty}^{\infty} e^{-i2\pi ux'}e^{A(s)+B(s)\nu_{t_0}}\left(\frac{\partial}{\partial x}A(s) + \frac{\partial}{\partial x}B(s)\nu_{t_0}\right)dudx'. \quad (4.0.7)
\end{aligned}
$$

$k < m$:

$$
\begin{aligned}
&\frac{\partial}{\partial x}\mathbb{P}\left[N_t = k \,|N_{t_0}\right] = \frac{\partial}{\partial x}\mathbb{P}\left[N_t \leq k+1|N_{t_0}\right] - \frac{\partial}{\partial x}\mathbb{P}\left[N_t \leq k|N_{t_0}\right] \\
&= \frac{\partial}{\partial x}G\left(k+1, 0, s, \nu_s\right) - \frac{\partial}{\partial x}G\left(k, 0, s, \nu_s\right) \\
&= \int_{-\infty}^{k+1}\int_{-\infty}^{\infty} e^{-i2\pi ux'}e^{A(s)+B(s)\nu_{t_0}}\left(\frac{\partial}{\partial x}A(s) + \frac{\partial}{\partial x}B(s)\nu_{t_0}\right)dudx' - \\
&\int_{-\infty}^{k}\int_{-\infty}^{\infty} e^{-i2\pi ux'}e^{A(s)+B(s)\nu_{t_0}}\left(\frac{\partial}{\partial x}A(s) + \frac{\partial}{\partial x}B(s)\nu_{t_0}\right)dudx' \\
&= \int_{k}^{k+1}\int_{-\infty}^{\infty} e^{-i2\pi ux'}e^{A(s)+B(s)\nu_{t_0}}\left(\frac{\partial}{\partial x}A(s) + \frac{\partial}{\partial x}B(s)\nu_{t_0}\right)dudx'. \quad (4.0.8)
\end{aligned}
$$

The coefficient functions $\partial A(t)/\partial x$ and $\partial B(t)/\partial x$ in expressions (4.0.7) and (4.0.8) can be found from the system of ODEs (3.2.23)-(3.2.24):

$$
\frac{\partial}{\partial s}A_x(s) = \frac{\partial}{\partial x}(\kappa c B(s)), \quad (4.0.9)
$$

$$
\frac{\partial}{\partial s}B_x(s) = -\frac{\partial}{\partial x}(\kappa B(s)) + e^{i2\pi u + \delta l B(s)}\frac{\partial}{\partial x}(\delta l B(s)). \quad (4.0.10)
$$

For example, the sensitivity of the expected loss process value with respect to c at c_0, i.e., $\frac{\partial}{\partial x}\mathbb{E}[L^{(index)}(t)]|_{c=c_0}$ leads to the following system of differential equations:

$$
\frac{\partial}{\partial s}A_c(s)|_{c=c_0} = \kappa B(s) + \kappa c_0 B_c(s)|_{c=c_0}, \quad (4.0.11)
$$

$$
\frac{\partial}{\partial s}B_c(s)|_{c=c_0} = -\kappa B_c(s)|_{c=c_0} + e^{i2\pi u + \delta l B(s)}\delta l B_c(s)|_{c=c_0}. \quad (4.0.12)
$$

4.1 Example calibration of CDS index contracts

Let us next consider the calibration of the intensity diffusion process parameters $\theta = \{\kappa, c, \delta, \nu_0\}, \kappa \geq 0, c > 0, \delta \geq 0$, $\nu_0 > 0$ with the Markit iTraxx Europe MAIN s30 5y contract with maturity date of 20th December 2023. The index quotes on 27th May 2019 are as shown in Table 3:

Table 3: Markit iTraxx Europe MAIN s30 5y (20-Dec-2023) Ref 62				
Tranche	Bid	Ask	Delta	Quote
0-3%	39.0625	39.8125	9.2	percentage point upfront + 1% spread
3-6%	8.3750	8.8750	5.2	percentage point upfront + 1% spread
6-12%	115.5000	120.5000	2.5	Spread, 0% recovery
12-100%	20.3750	21.6250	0.48	Spread, 40% recovery

To perform calibration we assume one-factor intensity model with fixed parameter values $\kappa = 1$, $\delta = 1$, $\nu_0 = c$. For the calibration we solve the following equation

$$\text{Model}(t, \kappa = 1, \delta = 1, \nu_0 = c, c) = \text{Mid}(t), \tag{4.1.1}$$

where t is 27th May 2019. The expression (4.1.1) is solved with bisection. The expression of model implied spread S_M or upfront payment F_M, i.e., Model$(t, \kappa = 1, \delta = 2, \nu_0 = c, c)$, writes:

$$\begin{aligned} S_M(t_0, t_N) &= \frac{(1-R)\int_{t_0}^{t_N} Z(t_0, s)\mathbb{E}\left[dL^{(tranche)}(t, K_1, K_2)\right] - F}{\sum_{n=1}^{N} \Delta(t_{n-1}, t_n) Z(t_0, t_n) \mathbb{E}\left[1 - L^{(tranche)}(t_i, K_1, K_2)\right]}, \\ F_M &= (1-R)\int_{t_0}^{t_N} Z(t_0, s)\mathbb{E}\left[dL^{(tranche)}(t, K_1, K_2)\right] - \\ &\quad S(t_0, t_N)\sum_{n=1}^{N} \Delta(t_{n-1}, t_n) Z(t_0, t_n) \mathbb{E}\left[1 - L^{(tranche)}(t_i, K_1, K_2)\right]. \end{aligned} \tag{4.1.2}$$

In the Table 4, calibrated values of c are shown.

Table 4: Calibrated values of c

Tranche	Mid	Calibrated c	Model implied spread (S_M) / upfront payment (F_M)
0-100%	62.0000	0.7938	$S^{(model)} = 61.9455$
0-3%	39.4375	0.4108	$F^{(model)} = 39.4371$
3-6%	8.6250	0.6499	$F^{(model)} = 8.6248$
6-12%	118.0000	0.5162	$S^{(model)} = 118.0460$
12-100%	21.0000	1.8301	$S^{(model)} = 20.1253$

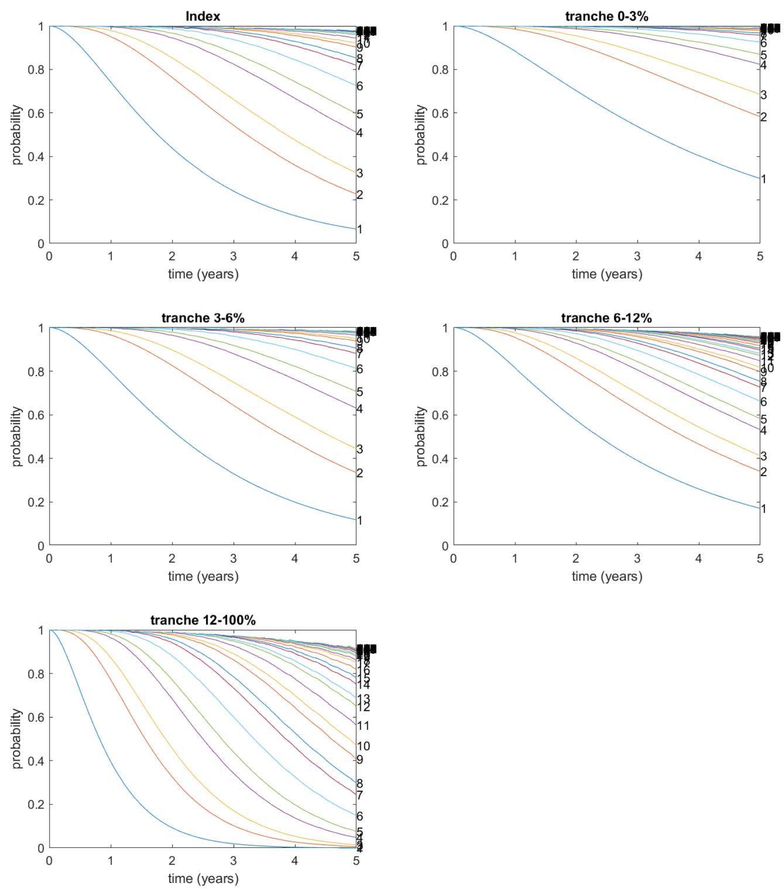

Figure 10: Model implied probability curves $\mathbb{P}[N_t \leq x]$. The number at right side of figure is the value of x. The curves are obtained with intensity process parameter values, $\kappa = 1$, $\delta = 1$, $\nu_0 = c$, where c is obtained from Table 4.

4.2 Calibration of the random thinning matrix

In order to random thin the default counting process $N(t)$, we need to specify the thinning matrix $\mathbf{M} \in \mathbb{R}^{m\times m}$, where $M_t^{kn} = \mathbb{P}\left[\tau^k = T^n | \mathcal{F}_{t-}\right]$. One straightforward procedure to address this problem is to choose elements M_t^{kn} such that the thinned probabilities $\mathbb{P}\left[t < \tau^k \leq T\right]$ yield the single-name CDS flat spreads equal to the market implied flat spread. In [6], computationally efficient method for solving elements M_t^{kn} via quadratic programming is derived. In this subsection, we summarize the method.

If we assume that the possible protection payment is done at the end of the current protection period, and we neglect the accrued interest payments, according to the expressions (2.1.1) and (2.1.7), we can approximate the value of the protection leg and premium leg as follows:

$$D^k = l \sum_{p=1}^{N} Z\left(t_0, t_p\right) \mathbb{P}\left[t_{p-1} < \tau^k \leq t_p\right] \tag{4.2.1}$$

$$P^k(S^k) = S^k \sum_{p=1}^{N} \Delta\left(t_{p-1}, t_p\right) Z\left(t_0, t_p\right) \mathbb{P}\left[t_p < \tau^k\right] \tag{4.2.2}$$

where D^k and P^k are the values of the protection and premium legs, respectively. The flat spread is found by equating (4.2.1) and (4.2.2):

$$\begin{aligned}
& l \sum_{p=1}^{N} Z\left(t_0, t_p\right) \mathbb{P}\left[t_{p-1} < \tau^k \leq t_p\right] = S^k \sum_{p=1}^{N} \Delta\left(t_{p-1}, t_p\right) Z\left(t_0, t_p\right) \left(1 - \mathbb{P}\left[0 < \tau^k \leq t_p\right]\right) \\
& \Rightarrow S^k \sum_{p=1}^{N} \Delta\left(t_{p-1}, t_p\right) Z\left(t_0, t_p\right) = l \sum_{p=1}^{N} Z\left(t_0, t_p\right) \mathbb{P}\left[t_{p-1} < \tau^k \leq t_p\right] + \\
& S^k \sum_{p=1}^{N} \Delta\left(t_{p-1}, t_p\right) Z\left(t_0, t_p\right) \mathbb{P}\left[t_0 < \tau^k \leq t_p\right].
\end{aligned} \tag{4.2.3}$$

We can write $\mathbb{P}\left[t < \tau^k \leq T\right] = -\sum_{n=1}^{m} M^{kn} \int_t^T \frac{\partial}{\partial z} \varphi_t\left(n-1, z, s, \lambda_s\right)|_{z=0} ds$ for time independent thinning matrix element M^{kn}, according to (3.2.12) and (3.2.13). This allows to write (4.2.3) as:

$$S^k W = \sum_{n=1}^{m} M^{kn} \left(V^n S^k + l X^n\right), \tag{4.2.4}$$

where

$$W=\sum_{p=1}^{N}\Delta\left(t_{p-1},t_p\right)Z\left(t_0,t_p\right), \tag{4.2.5}$$

$$V^n=-\sum_{p=1}^{N}\Delta\left(t_{p-1},t_p\right)Z\left(t_0,t_p\right)\int_{t_0}^{t_p}\frac{\partial}{\partial z}\varphi_t\left(n-1,z,s,\lambda_s\right)|_{z=0}ds, \tag{4.2.6}$$

$$X^n=-\sum_{p=1}^{N}Z\left(t_0,t_p\right)\int_{t_{p-1}}^{t_p}\frac{\partial}{\partial z}\varphi_t\left(n-1,z,s,\lambda_s\right)|_{z=0}ds. \tag{4.2.7}$$

To find random thinning matrix $\mathbf{M}$, we minimize the sum of the differences in model implied spreads and market implied spreads with the constraints that $M_t^{kn}\geq 0,\forall k,n$, and row and column sums of $\mathbf{M}$ equal to one. This leads to a following minimization problem:

$$\mathcal{P}_1:\begin{cases}\min\limits_{\mathbf{M}} & \sum_{k=1}^{m}\left(Mid(k)W-\sum_{n=1}^{m}M^{kn}\left(V^nMid(k)+lX^n\right)\right)^2 \quad (4.2.8a)\\ \text{s.t. :} & \sum_{k=1}^{m}M^{kn}=1,\forall n\in\{1,2,...,m\}, \quad (4.2.8b)\\ & \sum_{n=1}^{m}M^{kn}=1,\forall k\in\{1,2,...,m\}, \quad (4.2.8c)\\ & M^{kn}\geq 0,\forall k,n\in\{1,2,...,m\}, \quad (4.2.8d)\end{cases}$$

where $Mid(k)$ is the mid-quote of the flat spread for the k'th index constituent. By writing $a^{kn}=(V^nMid(k)+lX^n)$, $b^k=Mid(k)W$ and $c^{kn}=M^{kn}\left(V^nMid(k)+lX^n\right)$, the objective function (4.2.8a) becomes:

$$\sum_{k=1}^{m}\left(b^k-\sum_{n=1}^{m}c^{kn}\right)^2=\sum_{k=1}^{m}\left((b^k)^2-2b^k\sum_{n=1}^{m}c^{kn}+(\sum_{n=1}^{m}c^{kn})^2\right). \tag{4.2.9}$$

Therefore, the minimization problem $\mathcal{P}_1$ is equal to the following minimization problem:

$$\mathcal{P}_2:\begin{cases}\min\limits_{\mathbf{C}} & \sum_{k=1}^{m}(\sum_{n=1}^{m}c^{kn})^2-2\sum_{k=1}^{m}b^k\sum_{n=1}^{m}c^{kn} \quad (4.2.10a)\\ \text{s.t. :} & \sum_{k=1}^{m}\frac{c^{kn}}{a^{kn}}=1,\forall n\in\{1,2,...,m\}, \quad (4.2.10b)\\ & \sum_{n=1}^{m}\frac{c^{kn}}{a^{kn}}=1,\forall k\in\{1,2,...,m\}, \quad (4.2.10c)\\ & \frac{c^{kn}}{a^{kn}}\geq 0,\forall k,n\in\{1,2,...,m\}, \quad (4.2.10d)\end{cases}$$

Minimization problem $\mathcal{P}_2$ can be written in the matrix form as follows:

$$\mathcal{P}_3 : \begin{cases} \min\limits_{\mathbf{c}} & \frac{1}{2}\mathbf{c}^T\mathbf{R}\mathbf{c} + \mathbf{b}^T\mathbf{c} & \text{(4.2.11a)} \\ \text{s.t. :} & \mathbf{A}\mathbf{c} = \mathbf{e}, & \text{(4.2.11b)} \\ & \mathbf{B}\mathbf{c} = \mathbf{e}, & \text{(4.2.11c)} \\ & \mathbf{c} \succeq \mathbf{0}. & \text{(4.2.11d)} \end{cases}$$

where $\mathbf{c} \in \mathbb{R}^{m^2\times 1}$ is a vector of columns in $\mathbf{C}$, $\mathbf{R} \in \mathbb{R}^{m^2\times m^2}$ is a block diagonal matrix with diagonal unit matrices $\mathbf{E} \in \mathbb{R}^{m\times m}$, $\mathbf{b} \in \mathbb{R}^{m^2\times 1}$, $\mathbf{A} \in \mathbb{R}^{m^2\times m^2}$, $\mathbf{B} \in \mathbb{R}^{m^2\times m^2}$, $\mathbf{e} \in \mathbb{R}^{m^2\times 1}$ and $\mathbf{1} \in \mathbb{R}^{m\times 1}$ are vectors of ones:

$$\mathbf{R} = \begin{bmatrix} \mathbf{E} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{E} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{E} \end{bmatrix}, \mathbf{E} = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} \mathbf{C}_{(:,1)} \\ \mathbf{C}_{(:,2)} \\ \vdots \\ \mathbf{C}_{(:,m)} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \\ \vdots \\ \mathbf{b}^m \end{bmatrix}, \mathbf{b}^k = \begin{bmatrix} b^k \\ \vdots \\ b^k \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{1}^T & \mathbf{0}^T & \dots & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{1}^T & \dots & \mathbf{0}^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^T & \mathbf{0}^T & \dots & \mathbf{1}^T \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} \mathbf{1} & \overbrace{\mathbf{0} \ \dots \ \mathbf{0}}^{m-1 \text{ zeros}} & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \ \ \mathbf{0} \ \ \dots & \mathbf{0} & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots \ \ \ddots \ \ \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \dots \ \ \mathbf{0} \ \ \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

Note that matrix $\mathbf{R}$ is not full rank, and therefore does not have unique solution. To circumvent this problem, let us upper-bound the objective function (4.2.11a) by using matrix $\mathbf{Q} = \mathbf{R} + \mathbf{I}$, where $\mathbf{I}$ is identity matrix. Notice that $\mathbf{Q}$ is positive definite full rank matrix. Thus we have the final minimization problem that we address to find random thinning matrix $\mathbf{M}$:

$$\mathcal{P}_4 : \begin{cases} \min\limits_{\mathbf{c}} & \frac{1}{2}\mathbf{c}^T\mathbf{Q}\mathbf{c} + \mathbf{b}^T\mathbf{c} & \text{(4.2.12a)} \\ \text{s.t. :} & \mathbf{A}\mathbf{c} = \mathbf{e}, & \text{(4.2.12b)} \\ & \mathbf{B}\mathbf{c} = \mathbf{e}, & \text{(4.2.12c)} \\ & \mathbf{c} \succeq \mathbf{0}. & \text{(4.2.12d)} \end{cases}$$

Minimization problem $\mathcal{P}_4$ is a standard convex quadratic programming problem that can be solved with approximately cubic time-complexity (i.e., $O(n^3)$, where n is the problem dimension).

5 Pricing credit derivatives

In this section we price credit derivatives based on the methods derived in the sections 3 and 4. As noted in Section 2, the pricing of any credit derivative boils down to determining the probability of default in the derivative underlying. With this probability we can determine the expected payment flows of the product, and hence the value of the product.

The pricing with the default counting process models considered in Section 3 is a three step process. Firstly, we need to define the model, this means choosing the intensity process ν_t for the default counting process compensator. Secondly, we need to choose the calibration scheme to calibrate the free parameters of the model. The calibration is usually done based on the market observed quotes of calibration products, and the optimal parameters are obtained by solving an optimization problem or solving an equation with some root-finding solver. Thirdly, we need to apply the calibrated model to the product on hand. This means determining the payment flows via formulas discussed in Section 2, where the default probabilities are determined by the calibrated default counting process model.

In the following subsections, we price CDS index tranches and CDS index options with the default counting process model from Section 3. In the pricing of CDS index tranches, we calibrate the model to the market quotes of the whole index. We compare the model implied prices to market observed quotes, and try to explain the possible differences in prices. In the pricing of CDS index options, we focus on determining the sensitivities of the option contract with the default counting process models. This means calculating how much the option value changes based on changes in fair spread and fair spread volatility.

5.1 Pricing CDS index tranches

In this subsection we aim to value CDS index tranches based on the CDS index quotes, and compare the model implied prices to the observed market quotes on these tranches. In subsection 2.3, the valuation of tranched index contracts is considered, and the protection and premium legs write:

$$\mathbb{E}\left[\mathrm{PV}^{(\text{protection})}_{\text{CDS,tranche}}\left(t_0, t_N\right)\right] = (1-R)\int_{t_0}^{t_N} Z(t_0, s)\mathbb{E}\left[dL\left(t, K_1, K_2\right)\right], \tag{5.1.1}$$

$$\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{CDS,tranche}}(t_0, t_N)\right] = F + S\left(t_0, t_N\right)\sum_{n=1}^{N}\Delta\left(t_{n-1}, t_n\right) Z\left(t_0, t_n\right)\mathbb{E}\left[1 - L\left(t_i, K_1, K_2\right)\right], \tag{5.1.2}$$

where expected loss $\mathbb{E}\left[L\left(t, K_1, K_2\right)\right] = \frac{\max\{\mathbb{E}[N_t]-K_1,0\}-\max\{\mathbb{E}[N_t]-K_2,0\}}{K_2-K_1}$.

We choose to use intensity process ν_t according to (3.2.21). Moreover, we set all other parameters fixed except long-term equilibrium c. The fixed parameter values are $\kappa = 1.0$, $\delta = 1.0$ and $l = 0.6$. The calibration is performed as a root-finding exercise where the model implied price is equated with a market observed quote of the calibration product on given time instance. We use bisection to solve the root-finding problem.

We aim to value tranches of iTraxx Europe Crossover s30 5y index. The calibration product is the whole index. In Table 5, the quoting conventions for the index and the tranches are summarized:

Table 5: Markit iTraxx Europe Crossover s30 5y quoting convention

Tranche	Quote
index	Current flat premium leg spread $S_F(t, t_N)$.
0-10%	percentage point upfront + 5% spread.
10-20%	percentage point upfront + 5% spread.
20-35%	percentage point upfront + 5% spread.

The valuation is done on one of the business days in each week starting from 25.9.2018 and ending 19.9.2019. On each valuation date, we first determine c by solving equation $Mid(t) = Model(t)$, where $Mid(t)$ is market observed quote on iTraxx Europe Crossover s30 5y index on valuation date t and $Model(t)$ is the model implied price. After solving for c, we use the model to price tranches in Table 5 on the given valuation date.

In Figure 11, the market observed quotes and model implied values are plotted on the whole index and on the tranches in Table 5. We see that the market observed index quotes match perfectly with the model implied prices, since the index was the calibration product. On the 0%-10% tranche, model implied and market observed price profile is very similar but there is significant offset on the upfront. Then again, on the 10%-20% and 20%-35% tranches the market observed and model implied price profiles are substantially different.

The difference in the market observed and model implied prices boil down to several factors. Firstly, the market standard is to use the copula-framework to value tranches, whereas our approach is to use a default counting process model. This introduces model difference in the comparison. Secondly, our model is only allowing one variable to vary, namely long-term equilibrium c. This simplifies the model but at the same time makes the model not accurate if price depends on several factors. This is a common drawback in so-called one-factor models.

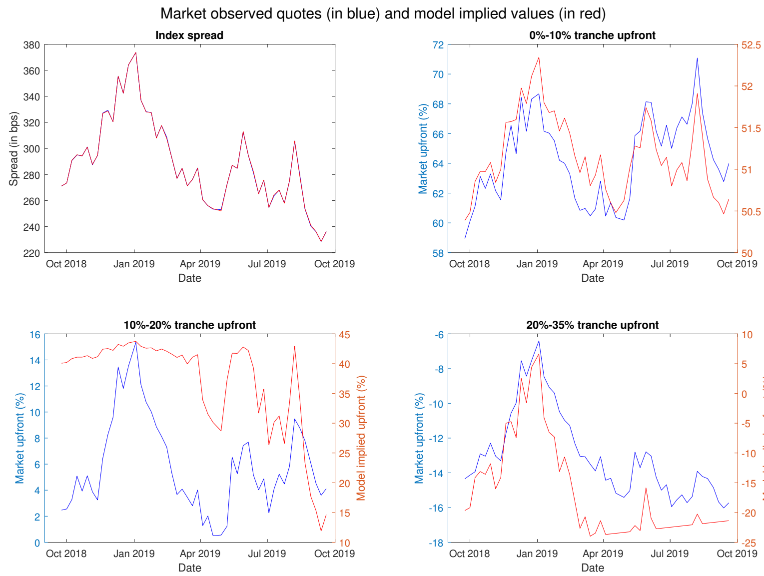

Figure 11: iTraxx Crossover s30 5y market observed and model implied tranche quotes when model is calibrated with the whole index.

5.2 Pricing CDS options

Let us next consider pricing CDS options with the model discussed in subsection 3.2. CDS options are important instruments in the credit derivative market due to their beneficial properties for the investors, such as possibility to take a position in the volatility of the credit market, and introducing non-linear payoff profiles for credit investing which can be combined to create desired payoff profiles.

In subsection 2.4. the pricing of the CDS index options is considered in detail. In Figure 11, the payment flows of European CDS index options are summarized. The payoff components are written in expressions (2.4.2), (2.4.4) and (2.4.5) and summarized below:

$$\begin{cases} FEP = \sum_{k=1}^{m} \mathbb{E}\left[\mathbf{1}_{\{\tau^k \le t_E\}}(1-R^k)\right], & (5.2.1a) \\ G(K) = \frac{K - S(t_E,T)}{S(t_E,T)}\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_E,T)\right], & (5.2.1b) \\ VC(t_E,T) = \mathbb{E}\left[\mathrm{PV}^{(\text{protection})}_{\text{CDS,index}}(t_E,T)\right] - \mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_E,T)\right]. & (5.2.1c) \end{cases}$$

Note that in expression (5.2.1c) the value of the protection leg equals to the

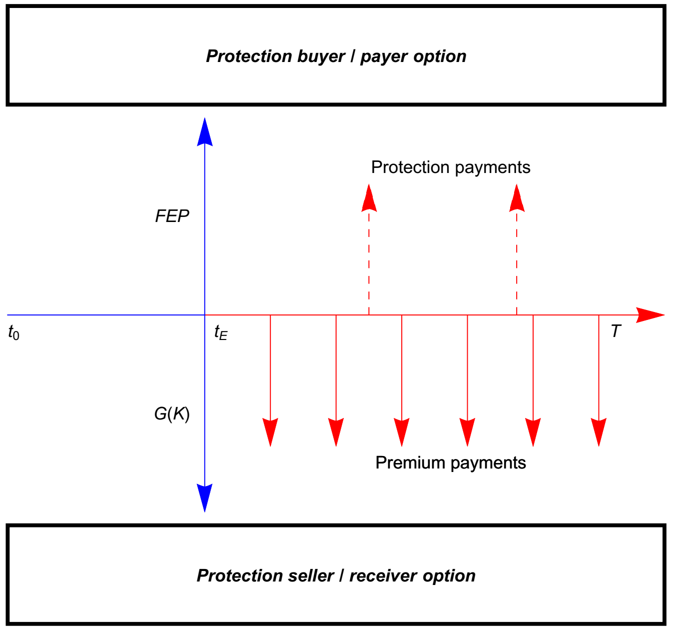

Figure 12: European CDS index option contract, with inception date t_0, exercise date t_E, and maturity date T.

value of the premium leg with flat spread $S_F(t_E, T)$:

$$\mathbb{E}\left[\mathrm{PV}^{(\text{protection})}_{\text{CDS,index}}(t_E, T)\right] = \mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_E, T; S_F(t_E, T))\right], \tag{5.2.2}$$

which gives us:

$$VC(t_E, T) = \mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_E, T; S_F(t_E, T))\right] - \mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_E, T)\right]. \tag{5.2.3}$$

To value the components (5.2.1a), (5.2.1b) and (5.2.3), we proceed as follows. The first step is to calculate the expected number of credit events in the front-end-protection period with the probability curves $\mathbb{P}[N_t \leq x]$ that are obtained by calibrating the given intensity process ν_t of the default counting process N_t to current flat spread $S_F(t_0, T)$ of the CDS contract. This gives us FEP.

The second step is to calibrate the intensity process ν_t to the flat spread at the exercise date t_E. To do this we need to assume some diffusion process for flat spread $S_F(t) = S_F(t, T)$. Natural first-step choices are either log-normal or normal dynamics. The log-normal dynamics give us:

$$\frac{dS_F(t)}{S_F(t)} = \mu dt + \sigma dB(t), \tag{5.2.4}$$

where $B(t)$ is a standard Brownian motion with $B(t_0) = 0$. The solution of the SDE (5.2.4) is $S_F(t) = S_F(t_0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)(t-t_0)+\sigma B(t)}$ (see [15]). By taking logarithm of the solution, we obtain:

$$\ln\left(S_F(t)\right) = \ln\left(S_F(t_0)\right) + \left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0) + \sigma \underbrace{B_t}_{\sim\mathcal{N}(0,t-t_0)} \tag{5.2.5}$$

which implies $\ln\left(S_F(t)\right) \sim \mathcal{N}\left(\bar{\mu}, \bar{\sigma}^2\right)$, where $\bar{\mu} = \ln\left(S_F(t_0)\right) + \left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0)$ and $\bar{\sigma}^2 = \sigma^2 * (t - t_0)$. Then again, the normal dynamics give us:

$$dS_F(t) = \mu dt + \sigma dB(t), \tag{5.2.6}$$

and by integrating expression (4.0.6), we get $S_F(t) = S_F(t_0) + \mu(t-t_0) + \sigma B(t)$. Therefore, we obtain $S_F(t) \sim \mathcal{N}\left(S_F(t_0) + \mu(t - t_0), \sigma^2 * (t - t_0)\right)$.

After we have chosen the dynamics for flat spread $S_F(t)$, we calibrate the dynamics to historical development of the flat spread, determine the expected value of flat spread at exercise date $\mathbb{E}\left[S_F(t_E)\right]$, and with that information calibrate the intensity process ν_{t_E}. This then gives us the expected value of the premium leg $\mathbb{E}\left[\mathrm{PV}^{(\text{premium})}_{\text{Non-Acc,index}}(t_E, T)\right]$.

In Figure 13, the value of European payer and receiver options with given spread bumps, i.e., additions to flat spread $S_F(t_E, T)$, are shown. The underlying swap contract is a CDS basket of 20 firms with running spread $S(t_E, T) = 4\%$ p.a. in premium leg, start date 13.03.2020 and end date 20.12.2023. The options have strike spread $K = 3.5\%$ p.a., start date 19.09.2019 and expiry date 13.03.2020. In this example, we use the normal dynamics of form (5.2.6) for the flat spread, with $S_F(t_0, T) = 1.86\%$. The spread bump profile describes the spread risk (i.e., delta) of the option contract.

In Figure 14, the value of European receiver option with volatility bumps, i.e., additions to volatility parameter σ in (5.2.6) are shown with flat spread $S_F(t_E, T) = 4.13\%$. The underlying swap contract is the same contract as in Figure 13. The volatility bump profile describes the volatility risk (i.e., vega) of the option contract.

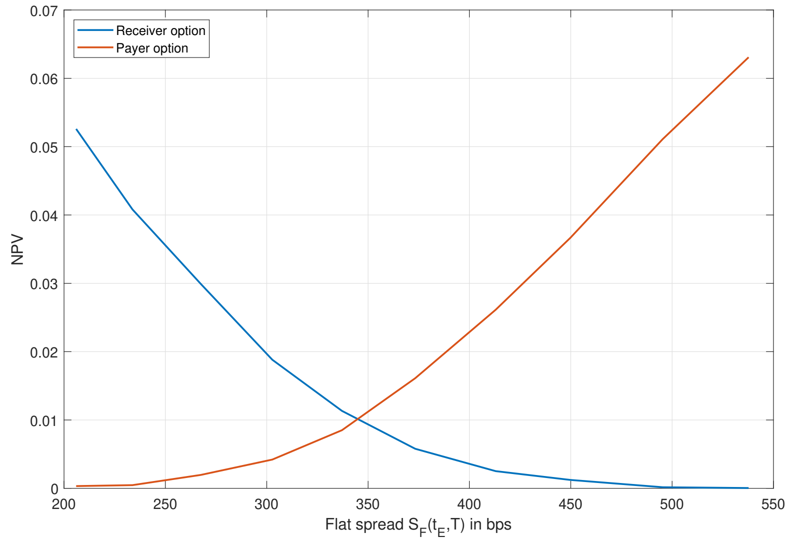

Figure 13: European payer and receiver CDS index option contracts with respect to bump on flat spread $S_F(t_E, T)$.

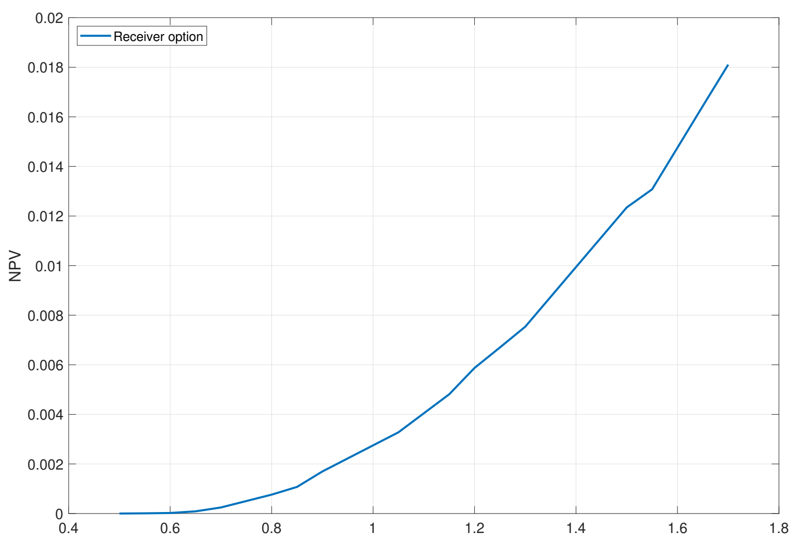

Figure 14: European payer and receiver CDS index option contracts with respect to bump on volatility σ.

6 Conclusion

In this thesis we introduced the reader to the pricing and modeling of various credit derivative products. We noted that the pricing boils down to determining the probability of defaults in the underlying. We focused on modeling this probability with a class of top-down models called generalized default counting process models. In this class of models, the modeling primitive is the point process that describes the number of credit events in the constituent firms of the underlying.

The most crucial modeling choice in the default counting process models was the choice of compensator intensity process. We introduced various affine intensity processes that can be considered in our modeling framework, and noted that these processes can address trends, mean-reversion, random disturbances and random jumps. When the intensity process was chosen from the class of affine processes, we were able to solve the needed default probabilities via a system of differential equations after a time-frequency transform.

In addition to the model derivation and implementation, we considered the calibration of the model. We discussed the situations where the calibration is done based on the solution of an optimization problem or a root-finding problem. For the optimization problem approach we considered gradient based methods, and for root-finding problems we chose to use bisection.

We assessed the model accuracy by pricing CDS index tranches with a single-factor default counting process model. The calibration was done with the quotes of the whole index, and we concluded that pricing of equity tranche was fairly accurate when the constant offset in a market quote and model implied price was excluded. The model implied risk numbers were examined with CDS index option pricing. We plotted the flat spread bump profiles (i.e., option delta) of European payer and receiver options, and the volatility bump profile (i.e., option vega) of receiver option.

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8 Appendices

8.1 Appendix A

In the derivation of the intensity based default counting process in subsection 3.2, we have to solve an expectation of form $\mathbb{E}\left[e^{i2\pi u H_t - z\nu_t}\right]$. To do this, let us write this expectation in an affine form:

$$\mathbb{E}\left[e^{i2\pi u H_t - z\nu_t}\right] = \mathbb{E}\left[e^{A(T-t)+B(T-t)\nu_t}\right], \tag{8.1.1}$$

where A and B are functions to be determined. To find the functions A and B, let us require that random variable $e^{A(T-t)+B(T-t)\nu_t}$ is a martingale (i.e., $e^{i2\pi u H_t - z\nu_t}$ is a martingale). This requirement yields:

$$\mathbb{E}\left[e^{A(T-t)+B(T-t)\nu_t}\right] = e^{A(T)+B(T)\nu_0}, \tag{8.1.2}$$

and accordingly $\mathbb{E}\left[e^{i2\pi u H_t - z\nu_t}\right] = e^{A(T)+B(T)\nu_0}$. Assume ν_t is a semimartingale of form:

$$d\nu_t = \mu(\nu_t)dt + \sigma(\nu_t)dW_t, \tag{8.1.3}$$

where W_t is a standard Brownian motion. Let us introduce function $f(t,x) = e^{A(T-t)+B(T-t)x}$. Clearly f is twice continuously differentiable. Thus, we are able to apply Itô's lemma:

$$\begin{aligned}
df(t,\nu_t) &= \sum_{i\in\{t,\nu_t\}} \frac{\partial f(t,\nu_t)}{\partial i} di + \frac{1}{2}\sum_{i,j\in\{t,\nu_t\}} \frac{\partial^2 f(t,\nu_t)}{\partial i \partial j} d\langle i,j\rangle_t \\
&= f(t,\nu_t)\left(-A'(T-t) - B'(T-t)\nu_t\right)dt + \\
&\quad f(t,\nu_t)B(T-t)d\nu_t + \frac{1}{2} f(t,\nu_t)B(T-t)^2 d\langle \nu,\nu\rangle_t \\
&\overset{*}{=} f(t,\nu_t)\left(-A'(T-t) - B'(T-t)\nu_t\right)dt + \\
&\quad f(t,\nu_t)B(T-t)\mu(\nu_t)dt + f(t,\nu_t)B(T-t)\sigma(\nu_t)dW_t \\
&\quad + \frac{1}{2} f(t,\nu_t)B(T-t)^2\sigma(\nu_t)^2 dt.
\end{aligned} \tag{8.1.4}$$

(*) Quadratic variation $d\langle \nu,\nu\rangle_t = \sigma(\nu_t)^2 d\langle W,W\rangle_t = \sigma(\nu_t)^2 dt$.

In order to $f(t,\nu_t)$ to be a martingale, the drift term in (8.1.4) needs to be zero. Therefore, we require:

$$B(T-t)\mu(\nu_t) + \frac{1}{2}B(T-t)^2\sigma(\nu_t)^2 = A'(T-t) + B'(T-t)\nu_t. \tag{8.1.5}$$

Based on the choices of $\mu(\nu_t)$ and $\sigma(\nu_t)$ in (8.1.3), the expression (8.1.5) leads to a system of differential equations.

For a jump diffusion process of form:

$$d\nu_t = \mu(\nu_t)dt + \sigma(\nu_t)dW_t + \delta(\nu_t)dZ_t, \tag{8.1.6}$$

where Z_t is a standard Poisson process with jump size of 1, Itô's lemma yields:

$$\begin{aligned} df(t,\nu_t) = f(t,\nu_t)\left(-A'(T-t) - B'(T-t)\nu_t\right)dt + \\ f(t,\nu_t)B(T-t)\mu(\nu_t)dt + f(t,\nu_t)B(T-t)\sigma(\nu_t)dW_t + \\ \frac{1}{2}f(t,\nu_t)B(T-t)^2\sigma(\nu_t)^2dt + \delta(\nu_t)f(t,\nu_t)dZ_t. \end{aligned} \tag{8.1.7}$$

The drift of the term $f(t,\nu_t)dZ_t$ can be written as $\lambda(\nu_t)\mathbb{E}\left[f(t,\nu_t+Z_t) - f(t,\nu_t)\right]$, where $\lambda(x)$ is the mean arrival rate of the jumps (see [8]). Thus, in order to $f(t,\nu_t)$ to be martingale, we require:

$$\begin{aligned} B(T-t)\mu(\nu_t) + \frac{1}{2}B(T-t)^2\sigma(\nu_t)^2 + \delta(\nu_t)\lambda(x)\mathbb{E}\left[f(t,\nu_t+Z_t) - f(t,\nu_t)\right] \\ = A'(T-t) + B'(T-t)\nu_t. \end{aligned} \tag{8.1.8}$$

Based on the choices of $\mu(\nu_t)$, $\sigma(\nu_t)$ and $\delta(\nu_t)$ in (8.1.6), the expression (8.1.8) leads to a system of differential equations.

8.2 Appendix B

In this appendix, we briefly consider the implementation of the method from subsection 3.2 to determine the default counting probabilities $\mathbb{P}[N_t \leq x]$. The implementation considered here is done with Matlab.

After the choice of intensity process ν_t, the first step is to solve the system of differential equations of form (3.2.23)-(3.2.24). For intensity process (3.2.21), the implementation to solve the system of ODEs could be as follows:

```
%% Solve system of ODEs:
%
u = (0:dt2:m).';      % Frequency parameter
scG = zeros(length(u),N1+1);
%
for k = 1:length(u)
    %The system of ODEs:
    f = @(t,x) [kappa*c*x(2);-kappa*x(2) - 1 + exp(1i*2*pi*u(k) + delta*l*x(2))];
    %
    % Solve ODEs and store the exp(A(s)+B(s)*v0):
    [t,xa] = ode45(f,[0:dt1:T],[0 -z]);
    scG(k,:) = exp(xa(:,1) +  xa(:,2)*v0).';
end
```

The idea above is to solve the system of ODEs for every frequency grid element u, and obtain a matrix scG which is expression (3.2.20) for given grid elements u. This is $\mathcal{G}$ in the expression (3.2.20).

Next step is to obtain G as in expression (3.2.16). This is done in two steps, first by applying time-frequency transform to $\mathcal{G}$ as in expression (3.2.25) and secondly by integrating over auxiliary variable x' as in expression (3.2.26). These steps can be done as follows:

```
%% Fourier transform to obtain the probabilities of default counter:
%
x_prime = (0:dt2:m);      % auxiliary variable for integration
G = zeros(length(x_prime),N1+1);
%
for k = 1:length(t)
    %
    % Fourier transform to obtain G:
    G(:,k) = 1/length(u)*fft(scG(:,k));
    %
    % Smoothen the function G:
    B = (abs(real(G(:,k))) > 10e-4);
```

```
    G(:,k) = real(G(:,k)).*B;
end


%% Integrating over the auxiliary variable x_prime:
%
% integrate over x_prime up to n to get the probability
    that default
% counter process at time s is less than or equal to n:
prob = zeros(m,length(t));
%
for n = 1:m
    for k = 1:length(t)
        prob(n,k) = sum(G((1:(n/dt2)),k));
    end
end
```

After these steps, we are ready to plot the default counting probabilities $\mathbb{P}[N_t \leq x]$. This can be done as follows:

```
%% Plot the figure of default counting process
    probabilities:
%
figure(1); clf;
plot(t,prob(1,:))
hold on
text(T,prob(1,end),num2str(1))
for k = 2:m
    plot(t,prob(k,:))
    text(T,prob(k,end),num2str(k))
end
xlabel('time (years)')
ylabel('probability')
hold off
```

The plotting produces a figure that looks as in Figure 4. The curves are dependent on the choices of free parameters in the intensity process.